

$$f(x) = \sum_{m=1}^4 p_m u(x_m)$$

$$= \frac{4}{9} \log x_1 + \frac{2}{9} \log x_2 + \frac{2}{9} \log x_3 + \frac{1}{9} \log x_4$$

$$g(x) = \sum_{m=1}^4 p_m \zeta_m x_m = ?$$

$$\zeta(\omega) = \frac{Z(\omega)}{(1+r)^2} = \frac{16}{25} Z(\omega) = \frac{16}{25} \cdot \frac{\tilde{P}(\omega)}{P(\omega)}, \quad p = \frac{2}{3}, \quad \tilde{p} = \frac{1}{2}$$

$$p_m \zeta_m = P(\omega^m) \zeta(\omega^m) = \frac{16}{25} \tilde{P}(\omega^m) = \frac{16}{25} \cdot \frac{1}{4} = \frac{4}{25} \quad \forall m$$

$$g(x) = \frac{4}{25} (x_1 + x_2 + x_3 + x_4)$$

$\nabla f(x) = \lambda \nabla g(x)$ becomes 4 eqns:

$$\left. \begin{aligned} \frac{\partial f}{\partial x_1} &= \lambda \frac{\partial g}{\partial x_1} \Rightarrow \frac{4}{9x_1} = \lambda \cdot \frac{4}{25} \Rightarrow x_1 = \frac{25}{9\lambda} \\ \frac{\partial f}{\partial x_2} &= \lambda \frac{\partial g}{\partial x_2} \Rightarrow \frac{2}{9x_2} = \lambda \cdot \frac{4}{25} \Rightarrow x_2 = \frac{25}{18\lambda} \\ \frac{\partial f}{\partial x_3} &= \lambda \frac{\partial g}{\partial x_3} \Rightarrow \frac{2}{9x_3} = \lambda \cdot \frac{4}{25} \Rightarrow x_3 = \frac{25}{18\lambda} \\ \frac{\partial f}{\partial x_4} &= \lambda \frac{\partial g}{\partial x_4} \Rightarrow \frac{1}{9x_4} = \lambda \cdot \frac{4}{25} \Rightarrow x_4 = \frac{25}{36\lambda} \end{aligned} \right\} (*)$$

Fifth eqn. is $g(x) = X_0$, or

$$\frac{4}{25} (x_1 + x_2 + x_3 + x_4) = 4$$

Sub (*) into this :

$$\frac{25}{4} \cdot \frac{4}{25} \left(\frac{25}{9\lambda} + \frac{25}{18\lambda} + \frac{25}{18\lambda} + \frac{25}{36\lambda} \right) = 4 \cdot \frac{25}{4}$$

$$\frac{100 + 50 + 50 + 25}{36\lambda} = \frac{225}{36\lambda} = 25 \Rightarrow 36\lambda = 9 \Rightarrow \lambda = \frac{1}{4}$$

$$\therefore x_1 = \frac{100}{9}, x_2 = \frac{50}{9}, x_3 = \frac{50}{9}, x_4 = \frac{25}{9}$$

$$X_2(\omega^m) = x_m$$

$$X_2(1, 1) = \frac{100}{9}$$

$$X_2(1, -1) = \frac{50}{9}$$

$$X_2(-1, 1) = \frac{50}{9}$$

$$X_2(-1, -1) = \frac{25}{9}$$

Step 2 Now replicate to find Δ . Same as before.

(III)

Approach the general optimal investment problem in this way:

$$\Omega = \Omega_N = \{\omega^1, \dots, \omega^M\}, \quad M = |\Omega| = 2^N$$

$$x_m = X_N(\omega^m), \quad p_m = P(\omega^m)$$

$$\zeta_m = \zeta(\omega^m) = \frac{Z(\omega^m)}{(1+r)^N} = \frac{1}{(1+r)^N} \cdot \frac{\tilde{P}(\omega^m)}{P(\omega^m)},$$

$$\text{so that } p_m \zeta_m = \frac{1}{(1+r)^N} \tilde{P}(\omega^m)$$

$$\text{Maximize: } E[U(X_N)] = \sum_{m=1}^M p_m U(x_m) =: f(x)$$

$$\begin{aligned} \text{Constraint: } X_0 &= \frac{1}{(1+r)^N} \tilde{E}[X_N] = \frac{1}{(1+r)^N} \sum_{m=1}^M x_m \tilde{P}(\omega^m) \\ &= \sum_{m=1}^M p_m \zeta_m x_m =: g(x) \end{aligned}$$

$$\frac{\partial f}{\partial x_m} = \lambda \frac{\partial g}{\partial x_m} \Rightarrow p_m U'(x_m) = \lambda p_m \zeta_m$$

$$\Rightarrow U'(x_m) = \lambda \zeta_m \Rightarrow U'(X_N) = \lambda \zeta \leftarrow \text{solve for } X_N$$

$$I = (U')^{-1}, \quad X_N = I(\lambda \zeta)$$

plug into constraint:

$$X_0 = g(x) = E[\zeta X_N] = E[\zeta I(\lambda \zeta)] \leftarrow \text{solve for } \lambda$$

Plug λ into $X_N = I(\lambda \zeta)$ and replicate.

Using this idea, we will present a theorem about the soln to the optimal investment problem.

First, let's rephrase it in more common mathematical language.

Let $\mathcal{D}$ be the set of adapted processes $\Delta = (\Delta_0, \dots, \Delta_{N-1})$. Given $x \in \mathbb{R}$ and $\Delta \in \mathcal{D}$, let $X^{x, \Delta}$ be the wealth process generated by the trading strategy, (x, Δ) . Let

$$V_{\Delta}(x) = E[U(X_N^{x, \Delta})]$$

and

$$V(x) = \sup_{\Delta \in \mathcal{D}} V_{\Delta}(x).$$

The optimal investment problem is an example of an "optimal control problem". The process Δ is

our control. By adjusting it, we can adjust the "payoff function", $V_{\Delta}(x)$. The maximum possible payoff is given by the "value function", $V(x)$.

Our problem is, given X_0 , find $\Delta^* \in \mathcal{D}$ s.t.
 $V_{\Delta^*}(X_0) = V(X_0)$.

Our first observation was that we could optimize X_N instead. In other words, we changed the optimization problem:

If W is a r.v., let (x^W, Δ^W) be the unique replicating strategy that leads to $X_N = W$.

Note that $x^W = \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[W]$. For $x \in \mathbb{R}$, let

$\mathcal{W}(x)$ be the set of r.v.s W with $x^W = x$.

For $W \in \mathcal{W}(x)$, let

$$\tilde{V}_W(x) = \mathbb{E}[U(W)]$$

and

$$\tilde{V}(x) = \sup_{W \in \mathcal{W}(x)} \tilde{V}_W(x).$$

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In this problem, we seek $W^* \in \mathcal{W}(x)$ s.t.

$$\tilde{V}_{W^*}(x) = \tilde{V}(x).$$

Lemma 3.3.4 For any $x \in \mathbb{R}$, we have $V(x) = \tilde{V}(x)$.

If $V_{\Delta^*}(x) = V(x)$ and $W^* = X_N^{x, \Delta^*}$, then

$$\tilde{V}_{W^*}(x) = \tilde{V}(x). \text{ Conversely, if } \tilde{V}_{W^*}(x) = \tilde{V}(x)$$

and $\Delta^* = \Delta^{W^*}$, then $V_{\Delta^*}(x) = V(x)$.

Pf: Let $\Delta \in \mathcal{D}(x)$. Define $W_0 = X_N^{x, \Delta} \in \mathcal{W}(x)$. Then

$$\begin{aligned} V_{\Delta}(x) &= E[U(X_N^{x, \Delta})] = E[U(W_0)] \\ &= \tilde{V}_{W_0}(x) \leq \sup_{W \in \mathcal{W}(x)} \tilde{V}_W(x) = \tilde{V}(x). \end{aligned}$$

This is true for all $\Delta \in \mathcal{D}(x)$, so

$$V(x) = \sup_{\Delta \in \mathcal{D}(x)} V_{\Delta}(x) \leq \tilde{V}(x).$$

Now let $W \in \mathcal{W}(x)$. Note that $W = X_N^{x, \Delta^W}$, so that

$$\begin{aligned} \tilde{V}_W(x) &= E[U(W)] = E[U(X_N^{x, \Delta^W})] \\ &= V_{\Delta^W}(x) \leq \max_{\Delta \in \mathcal{D}} V_{\Delta}(x) = V(x). \end{aligned}$$

This is true for all $W \in \mathcal{W}(x)$, so

$$\tilde{V}(x) = \sup_{W \in \mathcal{W}(x)} \tilde{V}_W(x) \leq V(x).$$

Together, this shows $V(x) = \tilde{V}(x)$.

Now suppose $V_{\Delta^*}(x) = V(x)$ and let

$W^* = X_N^{x, \Delta^*}$. Then

$$\begin{aligned} \tilde{V}_{W^*}(x) &= E[U(W^*)] = E[U(X_N^{x, \Delta^*})] \\ &= V_{\Delta^*}(x) = V(x) = \tilde{V}(x). \end{aligned}$$

Next, suppose $\tilde{V}_{W^*}(x) = \tilde{V}(x)$ and let $\Delta^* = \Delta^{W^*}$.

Then $W^* = X_N^{x, \Delta^*}$, so

$$\begin{aligned} V_{\Delta^*}(x) &= E[U(X_N^{x, \Delta^*})] = E[U(W^*)] \\ &= \tilde{V}_{W^*}(x) = \tilde{V}(x) = V(x). \quad \square \end{aligned}$$

With this lemma in place, we can focus on solving the second optimization problem. Before doing that, though, we need more assumptions on our utility function, U .

We will assume that U is continuously

diff'ble, strictly increasing, strictly concave, and $a > -\infty$.

Let $m_\infty = U'(\infty) := \lim_{x \rightarrow \infty} U'(x) \in [0, \infty)$. Let

$m_a = U'(a) := \lim_{x \downarrow a} U'(x) \in (m_\infty, \infty]$. Then

$U': [a, \infty] \rightarrow [m_\infty, m_a]$ is a strictly decreasing bijection. Let $I: [m_\infty, m_a] \rightarrow [a, \infty]$ be

its inverse function. Extend I to $[0, \infty]$ by

defining $I(y) = a$ for $y > m_a$ and $I(y) = \infty$ for $y < m_\infty$.

Lemma N3.3.8

Let $a_0 = (1+r)^{-N}a$. For each $x \in (a_0, \infty)$,

there exists $\lambda \in (0, \infty)$ s.t. $E[\zeta I(\lambda \zeta)] = x$.

Moreover, for any such λ , we have $\lambda \zeta > m_\infty$.

Pf: For $\lambda \in [0, \infty]$, define $J(\lambda) = E[\zeta I(\lambda \zeta)]$. It

can be shown that since I is continuous,

J is continuous. Also, $J(0) = \infty$ and

$$I(\infty) = E[\xi I(\infty)] = a E[\xi] = a_0 E[Z] = a_0.$$

Hence, for any $x \in (a_0, \infty)$, there exists $\lambda \in (0, \infty)$ s.t. $J(\lambda) = x$.

Now suppose $\lambda \xi(\omega) \leq m_\infty$ for some ω . Then $I(\lambda \xi(\omega)) = \infty$. Since $P(\omega) > 0$, we have $J(\lambda) = E[\xi I(\lambda \xi)] = \infty$, a contradiction. $\square$

Lemma N3.3.9

will later use this
with $y = \lambda \xi$

For any $y \in (m_\infty, \infty)$ and $c \in \mathbb{R}$,

$$U(I(y)) - y I(y) \geq U(c) - yc. \quad (*)$$

Pf: First, suppose $y \in (m_\infty, m_a)$. Then $y = U'(t)$ for some $t \in (a, \infty)$. Assume $c < t$. Then $\frac{U(t) - U(c)}{t - c} > U'(t)$, giving $U(t) - U(c) > U'(t)(t - c)$, which implies

$$U(t) - U'(t)t \geq U(c) - U'(t)c, \quad (**)$$

which is $(*)$. Next, assume $t < c$. Then

$$\frac{U(c) - U(t)}{c - t} < U'(t), \text{ giving } U(c) - U(t) < U'(t)(c - t),$$

which again implies $(**)$. The case $c = t$ implies $(**)$

trivially. Thus, (*) holds $\forall y \in (m_\infty, m_a)$.

Now suppose $y \geq m_a$. If $c \leq a$, then $U(c) = -\infty$, so (*) holds. We may therefore assume $c > a$.

By the limit of the previous case,

$$\begin{aligned} U(I(y)) - U(c) &= U(I(m_a)) - U(c) \\ &\geq m_a(I(m_a) - c) = m_a(I(y) - c). \end{aligned}$$

But $I(y) = a$ and $c > a$, so $I(y) - c$ is a negative real number. Thus,

$$U(I(y)) - U(c) \geq y(I(y) - c),$$

which is equivalent to (*). $\square$

Thm N3.3.10 (solution to optimal investment problem)

Let U be a utility function satisfying all of the assumptions in this section. Let $X_0 > a_0$ be given.

Let $\lambda > 0$ satisfy $J(\lambda) = E[\xi I(\lambda \xi)] = X_0$.

Define $W^* = I(\lambda \xi)$ and $\Delta^* = \Delta^{W^*}$. Then

$$V_{\Delta^*}(X_0) = V(X_0).$$

Pf: Note that

$$\begin{aligned} \frac{1}{(1+r)^N} \tilde{E}[W^*] &= E[\xi W^*] = E[\xi I(\lambda \xi)] \\ &= J(\lambda) = X_0. \end{aligned}$$

Thus, $W^* \in \mathcal{W}(X_0)$. By Lemma 3.3.4, it suffices to show that $\tilde{V}_{W^*}(X_0) = \tilde{V}(X_0)$. Let $W \in \mathcal{W}(X_0)$ be arbitrary. We want to show that $E[U(W^*)] \geq E[U(W)]$.

By Lemma N3.3.8, $\lambda \xi > m_\infty$. We may therefore apply Lemma N3.3.8 with $y = \lambda \xi$ and $c = W$ to get

$$U(W^*) - \lambda \xi W^* \geq U(W) - \lambda \xi W.$$

Thus,

$$\begin{aligned} E[U(W^*) - U(W)] &= E[U(W^*) - \lambda \xi W^* - (U(W) - \lambda \xi W)] \\ &\quad + E[\lambda \xi (W^* - W)] \\ &\geq E[\lambda \xi (W^* - W)] \\ &= \lambda \cdot \frac{1}{(1+r)^N} \tilde{E}[W^* - W] \\ &= \lambda (X_0 - X_0) = 0. \quad \square \end{aligned}$$

Expl 3.3.2 again (using theorem)

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\frac{2}{3}$$

$$U(x) = \log x, \quad X_0 = 4$$

Find Δ that maximizes $E[U(X_2)]$.

$$a = 0 > -\infty \checkmark$$

U is strictly decr., strictly concave, cont. diff'ble on $(0, \infty) \checkmark$

$$U'(x) = \frac{1}{x} \text{ for } x \in (0, \infty)$$

$$m_0 = U'(0) = \infty, \quad m_\infty = U'(\infty) = 0$$

$$\begin{aligned} I(y) &= (U')^{-1}(y) : [m_\infty, m_0] \rightarrow [a, \infty] \\ &\quad : [0, \infty] \rightarrow [0, \infty] \end{aligned}$$

$$y = U'(x) = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$$I(y) = \frac{1}{y} \text{ for } y \in (0, \infty)$$

$$I(\infty) = 0, \quad I(0) = \infty, \quad I(y) = \infty \quad \forall y < 0$$

$$a_0 = \frac{1}{(1+r)^N} a = 0$$

$$X_0 = 4 > a_0 \checkmark$$

Solve $E[\xi I(\lambda \xi)] = 4$ for $\lambda > 0$ (a soln exists)