

$$\Delta = (\Delta_0, \Delta_1), \quad \Delta_1 = \Delta_1(\xi_1).$$

Need to find 3 numbers: $\Delta_0, \Delta_1(1), \Delta_1(-1)$

Plan: Step 1: Find the maximum value, X_2 .

Step 2: Find the maximizer, Δ .

Step 1

$$X_{n+1} = (1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}$$

$$X_1 = \frac{5}{4}(4 - 4\Delta_0) + S_1 \Delta_0$$

$$= 5 - 5\Delta_0 + S_1 \Delta_0 = (S_1 - 5)\Delta_0 + 5$$

$$X_1(1) = 3\Delta_0 + 5, \quad X_1(-1) = -3\Delta_0 + 5$$

$$\begin{aligned} X_2 &= \frac{5}{4}(X_1 - \Delta_1 S_1) + \Delta_1 S_2 \\ &= \Delta_1(S_2 - \frac{5}{4}S_1) + \frac{5}{4}(\overbrace{(S_1 - 5)\Delta_0 + 5}^{X_1}) \\ &= \Delta_1(S_2 - \frac{5}{4}S_1) + \frac{5}{4}(S_1 - 5)\Delta_0 + \frac{25}{4} \end{aligned}$$

$$X_2(1,1) = 6\Delta_1(1) + \frac{15}{4}\Delta_0 + \frac{25}{4}$$

$$X_2(1,-1) = -6\Delta_1(1) + \frac{15}{4}\Delta_0 + \frac{25}{4}$$

$$X_2(-1,1) = \frac{3}{2}\Delta_1(-1) - \frac{15}{4}\Delta_0 + \frac{25}{4}$$

$$X_2(-1,-1) = -\frac{3}{2}\Delta_1(-1) - \frac{15}{4}\Delta_0 + \frac{25}{4}$$

Define

$$f_1(x, y, z) = 6y + \frac{15}{4}x + \frac{25}{4}$$

$$f_2(x, y, z) = -6y + \frac{15}{4}x + \frac{25}{4}$$

$$f_3(x, y, z) = \frac{3}{2}z - \frac{15}{4}x + \frac{25}{4}$$

$$f_4(x, y, z) = -\frac{3}{2}z - \frac{15}{4}x + \frac{25}{4}$$

Then

$$X_2(1, 1) = f_1(\Delta_0, \Delta_1(1), \Delta_1(-1))$$

$$X_2(1, -1) = f_2(\Delta_0, \Delta_1(1), \Delta_1(-1))$$

$$X_2(-1, 1) = f_3(\Delta_0, \Delta_1(1), \Delta_1(-1))$$

$$X_2(-1, -1) = f_4(\Delta_0, \Delta_1(1), \Delta_1(-1))$$

Define $f := \frac{4}{9} \log f_1 + \frac{2}{9} \log f_2 + \frac{2}{9} \log f_3 + \frac{1}{9} \log f_4$.

Then

$$E[u(X_2)] = E[\log X_2]$$

$$= \frac{4}{9} \log X(1, 1) + \frac{2}{9} \log X(1, -1) + \frac{2}{9} \log X(-1, 1) + \frac{1}{9} \log X(-1, -1)$$

$$= f(\Delta_0, \Delta_1(1), \Delta_1(-1))$$

Want to maximize $f(x, y, z)$.

Find critical points where $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = 0$.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{4}{9} \cdot \frac{1}{f_1} \cdot \frac{\partial f_1}{\partial x} + \frac{2}{9} \cdot \frac{1}{f_2} \cdot \frac{\partial f_2}{\partial x} + \frac{2}{9} \cdot \frac{1}{f_3} \cdot \frac{\partial f_3}{\partial x} + \frac{1}{9} \cdot \frac{1}{f_4} \cdot \frac{\partial f_4}{\partial x} \\ &= \frac{5}{12} \left(\frac{4}{f_1} + \frac{2}{f_2} - \frac{2}{f_3} - \frac{1}{f_4} \right) = 0 \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{4}{9} \cdot \frac{1}{f_1} \cdot \frac{\partial f_1}{\partial y} + \frac{2}{9} \cdot \frac{1}{f_2} \cdot \frac{\partial f_2}{\partial y} \\ &= \frac{4}{3} \left(\frac{2}{f_1} - \frac{1}{f_2} \right) = 0 \end{aligned} \quad (2)$$

$$\frac{\partial f}{\partial z} = \frac{2}{9} \cdot \frac{1}{f_3} \cdot \frac{\partial f_3}{\partial z} + \frac{1}{9} \cdot \frac{1}{f_4} \cdot \frac{\partial f_4}{\partial z} = \frac{1}{6} \left(\frac{2}{f_3} - \frac{1}{f_4} \right) = 0 \quad (3)$$

$$(2) \Rightarrow \boxed{f_1 = 2f_2}$$

$$(3) \Rightarrow \boxed{f_3 = 2f_4}$$

$$\text{sub into (1)} \Rightarrow \frac{4}{f_2} - \frac{2}{f_4} = 0 \Rightarrow \boxed{f_2 = 2f_4}$$

Boxed eqns. give :

$$X_2(1,1) = 2 X_2(1,-1)$$

$$X_2(-1,1) = 2 X_2(-1,-1)$$

$$X_2(1,-1) = 2 X_2(-1,-1)$$

Trying to find X_2 . Have 3 eqns. in 4 unknowns.

Need a 4th eqn:

$\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -mart.

$$X_0 = \tilde{\mathbb{E}} \left[\frac{X_N}{(1+r)^N} \right]$$

$$4 = \left(\frac{4}{5} \right)^2 \tilde{\mathbb{E}}[X_2]$$

$$4 = \frac{16}{25} \left(\frac{1}{4} X_2(1,1) + \frac{1}{4} X_2(1,-1) + \frac{1}{4} X_2(-1,1) + \frac{1}{4} X_2(-1,-1) \right)$$

Solve the system of linear eqns...

$$X_2(1,1) = \frac{100}{9}$$

$$X_2(-1,1) = \frac{50}{9}$$

$$X_2(1,-1) = \frac{50}{9}$$

$$X_2(-1,-1) = \frac{25}{9}$$

So the optimal Δ will produce this X_2 ↗

Step 2

(106)

How to find Δ ? Replicate!

$$\Delta_1(1) = \frac{X_2(1,1) - X_2(1,-1)}{S_2(1,1) - S_2(1,-1)} = \frac{\frac{100}{9} - \frac{50}{9}}{16 - 4} = \frac{50}{9(12)} = \frac{25}{54}$$

$$\Delta_1(-1) = \frac{X_2(-1,1) - X_2(-1,-1)}{S_2(-1,1) - S_2(-1,-1)} = \frac{\frac{50}{9} - \frac{25}{9}}{4 - 1} = \frac{25}{9(3)} = \frac{25}{27}$$

$$\Delta_0 = \frac{X_1(1) - X_1(-1)}{S_1(1) - S_1(-1)} = ?$$

$$X_1(1) = \frac{1}{1+r} (\tilde{p} X_2(1,1) + \tilde{q} X_2(1,-1))$$

$$= \frac{2}{5} \cdot \frac{100}{9} + \frac{2}{5} \cdot \frac{50}{9} = \frac{40+20}{9} = \frac{60}{9} = \frac{20}{3}$$

$$X_1(-1) = \frac{2}{5} \cancel{\frac{50}{9} X_2(-1,1)} + \frac{2}{5} \cancel{\frac{25}{9} X_2(-1,-1)} = \frac{20+10}{9} = \frac{30}{9} = \frac{10}{3}$$

$$\Delta_0 = \frac{\frac{20}{3} - \frac{10}{3}}{8 - 2} = \frac{10}{3(6)} = \frac{5}{9}$$

Solution:

$$\Delta_0 = \frac{5}{9}, \quad \Delta_1 = \begin{cases} \frac{25}{54} & \text{if } \xi_1 = 1, \\ \frac{25}{27} & \text{if } \xi_1 = -1 \end{cases}$$

Exl 3.3.2 summary of methodStep 1

Find the r.v. X_N that maximizes $E[U(X_N)]$
subject to the constraint,

$$X_0 = \tilde{E} \left[\frac{X_N}{(1+r)^N} \right].$$

Step 2 Replicate X_N in order to find Δ .

Exl 3.3.2 again (more systematic)

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, q=\frac{2}{3}$$

$$U(x) = \log x, X_0=4$$

Find Δ that maximizes $E[U(X_2)]$.

Step 1

Find X_2 that

- maximizes $E[U(X_2)] = E[\log X_2]$

- constraint: $X_0 = \tilde{E} \left[\frac{X_2}{(1+r)^2} \right] = E \left[\frac{X_2 Z}{(1+r)^2} \right] = E[\zeta X_2]$

$$\Omega = \Omega_2 = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$$

$\uparrow$
 ω^1

$\uparrow$
 ω^2

$\uparrow$
 ω^3

$\uparrow$
 ω^4

looking for these

$$x_m = X_2(\omega^m), \quad p_m = P(\omega^m), \quad \zeta_m = \zeta(\omega^m)$$

$$E[U(X_2)] = \sum_{m=1}^4 p_m U(x_m)$$

$$E[\zeta X_2] = \sum_{m=1}^4 p_m \zeta_m x_m$$

Rewording of problem:

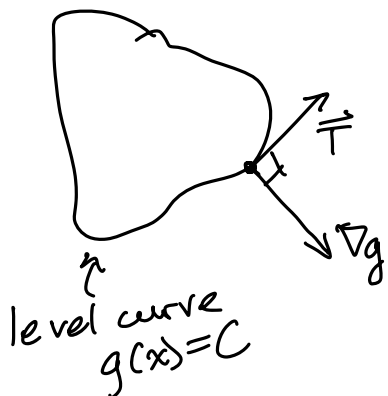
Find (x_1, x_2, x_3, x_4) that

maximizes: $\sum_{m=1}^4 p_m U(x_m)$

constraint: $\sum_{m=1}^4 p_m \zeta_m x_m = X_0$

Crash course on method of Lagrange multipliers:

Maximize $f(x)$ subject to $g(x) = C$ (x is a vector)



dir. deriv. of f in dir. T is $\nabla f \cdot \vec{T}$.

want $\nabla f \cdot \vec{T} = 0$, so want $\nabla f, \nabla g$ parallel.

Solve:

$$\begin{aligned} \nabla f(x) &= \lambda \nabla g(x) \\ g(x) &= C \end{aligned}$$

for x .

$$f(x) = \sum_{m=1}^4 p_m u(x_m)$$

$$= \frac{4}{9} \log x_1 + \frac{2}{9} \log x_2 + \frac{2}{9} \log x_3 + \frac{1}{9} \log x_4$$

$$g(x) = \sum_{m=1}^4 p_m \zeta_m x_m = ?$$

$$\zeta(\omega) = \frac{Z(\omega)}{(1+r)^2} = \frac{16}{25} Z(\omega) = \frac{16}{25} \cdot \frac{\tilde{P}(\omega)}{P(\omega)}, \quad p = \frac{2}{3}, \quad \tilde{p} = \frac{1}{2}$$

$$p_m \zeta_m = P(\omega^m) \zeta(\omega^m) = \frac{16}{25} \tilde{P}(\omega^m) = \frac{16}{25} \cdot \frac{1}{4} = \frac{4}{25} \quad \forall m$$

$$g(x) = \frac{4}{25} (x_1 + x_2 + x_3 + x_4)$$

$\nabla f(x) = \lambda \nabla g(x)$ becomes 4 eqns:

$$\left. \begin{aligned} \frac{\partial f}{\partial x_1} &= \lambda \frac{\partial g}{\partial x_1} \Rightarrow \frac{4}{9x_1} = \lambda \cdot \frac{4}{25} \Rightarrow x_1 = \frac{25}{9\lambda} \\ \frac{\partial f}{\partial x_2} &= \lambda \frac{\partial g}{\partial x_2} \Rightarrow \frac{2}{9x_2} = \lambda \cdot \frac{4}{25} \Rightarrow x_2 = \frac{25}{18\lambda} \\ \frac{\partial f}{\partial x_3} &= \lambda \frac{\partial g}{\partial x_3} \Rightarrow \frac{2}{9x_3} = \lambda \cdot \frac{4}{25} \Rightarrow x_3 = \frac{25}{18\lambda} \\ \frac{\partial f}{\partial x_4} &= \lambda \frac{\partial g}{\partial x_4} \Rightarrow \frac{1}{9x_4} = \lambda \cdot \frac{4}{25} \Rightarrow x_4 = \frac{25}{36\lambda} \end{aligned} \right\} (*)$$