

§3.1 Change of measure

Ω : finite sample space

$P, \tilde{P}$: prob measures on Ω

Assume $P(\omega) > 0$ and $\tilde{P}(\omega) > 0 \quad \forall \omega \in \Omega$.

$$Z(\omega) := \frac{\tilde{P}(\omega)}{P(\omega)}$$

↑ a random variable

called the Radon-Nikodym derivative
of $\tilde{P}$ with respect to P

Thm 3.1.1

(i) $P(Z > 0) = 1$

(iii) for any r.v. Y , $\tilde{E}[Y] = E[ZY]$

(iii') for any event A , $\tilde{P}(A) = E[Z1_A]$

(ii) $E[Z] = 1$

(iv) for any r.v. Y , $E[Y] = \tilde{E}[\frac{1}{Z}Y]$

(iv') for any event A , $P(A) = \tilde{E}[\frac{1}{Z}1_A]$

(ii') $\tilde{E}[\frac{1}{Z}] = 1$

Pf:

$$(i) \quad Z(\omega) > 0 \quad \forall \omega \in \Omega \Rightarrow P(Z > 0) = P(\Omega) = 1.$$

$$(iii) \quad \tilde{E}[Y] = \sum_{\omega \in \Omega} Y(\omega) \tilde{P}(\omega) = \sum_{\omega \in \Omega} Y(\omega) Z(\omega) P(\omega) = E[YZ].$$

$$(iii') \quad \text{Use } Y = 1_A \text{ in (iii).}$$

$$(ii) \quad \text{Use } Y = 1 \text{ in (iii)}$$

$$(iv), (iv'), (ii) : \text{Follows since } \frac{1}{Z} \text{ is the R-N deriv.}$$

of P w.r.t. $\tilde{P}$. $\square$

Expl

$$\text{Let } \Omega = \Omega_3, \quad p = \frac{2}{3}, \quad P = P^p, \quad \tilde{p} = \frac{1}{2}, \quad \tilde{P} = P^{\tilde{p}}.$$

Let Z be the R-N deriv. of $\tilde{P}$ w.r.t. P .

Find a formula for $Z(\omega)$ in terms of

$$\alpha(\omega) = |\{n : \omega_n = 1\}|. \quad \text{Compute } Z(-1, 1, 1).$$

$$P(\omega) = p^{\alpha(\omega)} q^{N-\alpha(\omega)} = \left(\frac{2}{3}\right)^\alpha \left(\frac{1}{3}\right)^{3-\alpha}$$

$$\tilde{P}(\omega) = \tilde{p}^{\alpha(\omega)} \tilde{q}^{N-\alpha(\omega)} = \left(\frac{1}{2}\right)^\alpha \left(\frac{1}{2}\right)^{3-\alpha} = \frac{1}{8}$$

$$Z(\omega) = \frac{\tilde{P}(\omega)}{P(\omega)} = \frac{1}{8} \left(\frac{3}{2}\right)^\alpha 3^{3-\alpha}$$

$$= \frac{27}{8} \left(\frac{3}{2}\right)^\alpha \left(\frac{1}{3}\right)^\alpha = \boxed{\frac{27}{8} \left(\frac{1}{2}\right)^\alpha}$$

$$Z(-1, 1, 1) = \frac{27}{8} \left(\frac{1}{2}\right)^{\alpha(-1, 1, 1)} = \frac{27}{8} \left(\frac{1}{2}\right)^2 = \boxed{\frac{27}{32}}$$

In general,

$$Z(\omega) = \frac{\tilde{P}(\omega)}{P(\omega)} = \frac{\tilde{p}^\alpha \tilde{q}^{N-\alpha}}{p^\alpha q^{N-\alpha}}$$

$$\boxed{Z(\omega) = \left(\frac{\tilde{q}}{q}\right)^N \left(\frac{\tilde{p}/\tilde{q}}{p/q}\right)^{\alpha(\omega)}}$$

odds
 ("odds of A" is $\frac{P(A)}{P(A^c)}$)

Expl

$$N=3, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\frac{2}{3}$$

$$M_n = \max \{S_0, \dots, S_n\}, \quad V_N = M_N - S_N.$$

Use Thm 3.1.1 (iii) to compute V_0 .

$$V_0 = \frac{1}{(1+r)^N} \tilde{E}[V_N] = \left(\frac{4}{5}\right)^3 \tilde{E}[V_3]$$

$$\stackrel{\text{Thm 3.1.1(iii)}}{\uparrow} \frac{64}{125} E[Z V_3] = \dots = \boxed{1.376}$$

see calculations
in book

Defn N3.1.1 In the binomial model, the state price density is $\zeta = \frac{Z}{(1+r)^N}$, where Z is the R-N deriv. of $\tilde{P}$ w.r.t. P . a r.v.

Thm N3.1.2

Fix $\omega^* \in \Omega$ and let $V_N = 1_{\{\omega^*\}}$.

Then $V_0 = \underbrace{\zeta(\omega^*) P(\omega^*)}_{\text{state price corresponding to } \omega^*}$

Intuition: ω^* is some fixed pattern of 1's and -1's.

(It's one of the 2^N possible "states of the world".)

V_N pays \$1 if ω^* is the true state (i.e. if the coin flips exactly match this pattern) and \$0 otherwise. V_0 is the "state price" of ω^* .

Pf: $V_0 = \frac{1}{(1+r)^N} \tilde{E}[V_N] = \frac{1}{(1+r)^N} \tilde{P}(\omega^*)$
 $= \frac{1}{(1+r)^N} Z(\omega^*) P(\omega^*) = Z_0(\omega^*) P(\omega^*) . \square$

Thm N3.1.3

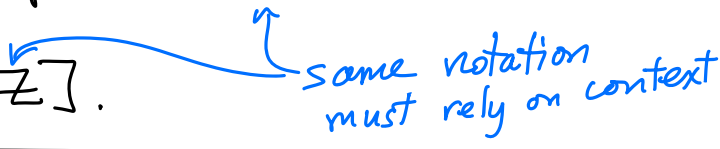
If V_N is a deriv. secur., then $V_0 = E[Z V_N]$.

Pf: $V_0 = \frac{1}{(1+r)^N} \tilde{E}[V_N] = \frac{1}{(1+r)^N} E[Z V_N] = E[Z V_N] . \square$

§3.2 Radon-Nikodym derivative process

Defn N3.2.1 The Radon-Nikodym derivative

process is the process $Z = (Z_0, \dots, Z_N)$ defined

by $Z_n = E_n[\underline{Z}]$.  same notation must rely on context

Note: $Z_N = E_N[Z] = Z$ and

$$Z_0 = E[Z] = 1$$

Thm 3.2.1

Z is a mart. under P .

Pf: $Z_n = E_n[Z]$ a function of $\xi_1, \dots, \xi_n$

$\therefore Z$ is adapted $\checkmark$

$$E_n[Z_{n+1}] = E_n[E_{n+1}[Z]] = E_n[Z] = Z_n. \checkmark \square$$

Expl 3.2.3

$$N=3, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\frac{2}{3}$$

(a) Find $Z_2(1,1)$ and $Z_2(1,-1)$.

(b) Use the defn of Z_1 to compute $Z_1(1)$.

(c) Use the mart. prop. to compute $Z_1(1)$.

$$(a) \quad Z = \left(\frac{\tilde{q}}{q}\right)^N \left(\frac{\tilde{p}/\tilde{q}}{p/q}\right)^\alpha, \quad \tilde{p} = \tilde{q} = \frac{1}{2}$$

$$Z = \frac{27}{8} \left(\frac{1}{2}\right)^\alpha$$

$$Z_2(1,1) = E_2[Z](1,1)$$

$$= E[Z | \xi_1=1, \xi_2=1]$$

$$= E[Z(1,1, \xi_3)]$$

$$= p Z(1,1,1) + q Z(1,1,-1)$$

(91)

$$\begin{aligned}
&= \frac{2}{3} \cdot \frac{27}{8} \left(\frac{1}{2}\right)^3 + \frac{1}{3} \cdot \frac{27}{8} \cdot \left(\frac{1}{2}\right)^2 \\
&= \frac{9}{32} + \frac{9}{32} = \boxed{\frac{9}{16}}
\end{aligned}$$

$$\begin{aligned}
Z_2(1, -1) &= E_2[Z](1, -1) \\
&= E[Z \mid \xi_1 = 1, \xi_2 = -1] \\
&= E[Z(1, -1, \xi_3)] \\
&= p Z(1, -1, 1) + q Z(1, -1, -1) \\
&= \frac{2}{3} \cdot \frac{27}{8} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{3} \cdot \frac{27}{8} \cdot \frac{1}{2} \\
&= \frac{9}{16} + \frac{9}{16} = \boxed{\frac{9}{8}}
\end{aligned}$$

$$\begin{aligned}
(b) \quad Z_1(1) &= E_1[Z](1) \\
&= E[Z \mid \xi_1 = 1] \\
&= E[Z(1, \xi_2, \xi_3)] \\
&= p^2 Z(1, 1, 1) + pq Z(1, 1, -1) \\
&\quad + qp Z(1, -1, 1) + q^2 Z(1, -1, -1) \\
&= \frac{27}{8} \left(\frac{4}{9} \cdot \left(\frac{1}{2}\right)^3 + 2 \cdot \frac{2}{9} \left(\frac{1}{2}\right)^2 + \frac{1}{9} \cdot \frac{1}{2} \right)
\end{aligned}$$

(92)

$$= \frac{27}{8} \left(\frac{1}{18} + \frac{1}{9} + \frac{1}{18} \right) = \frac{27}{8} \cdot \frac{2}{9}$$

$$= \boxed{\frac{3}{4}}$$

$$(c) \quad Z_1(1) = E_1[Z_2](1)$$

$$= E[Z_2 | \xi_1 = 1]$$

$$= E[Z_2(1, \xi_2)]$$

$$= p Z_2(1, 1) + q Z_2(1, -1)$$

$$= \frac{2}{3} \cdot \frac{9^3}{168} + \frac{1}{3} \cdot \frac{9^3}{8}$$

$$= \frac{3}{8} + \frac{3}{8} = \boxed{\frac{3}{4}}$$

Lemma 3.2.5

If Y depends only on $\xi_1, \dots, \xi_n$, then

$$\tilde{E}[Y] = E[Z_n Y].$$

Pf: $\tilde{E}[Y] = E[Z Y] = E[E_n[Z Y]]$

$$= E[Y \cancel{E_n[Z]}^{Z_n}] = E[Z_n Y]. \quad \square$$

(93)

Abuse of notation: If $\theta \in \{-1, 1\}^n$, ← little n ,
not big N

then $\alpha(\theta) = |\{j: \theta_j = 1\}|$.

For $\theta \in \{-1, 1\}^n$, let

$$A_\theta = \{\omega \in \Omega_N: \omega_j = \theta_j \text{ for } j \leq n\}$$

= "The first n flips match θ ."

Thm N3.2.2

$$\text{For all } \theta \in \{-1, 1\}^n, \quad Z_n(\theta) = \frac{\tilde{P}(A_\theta)}{P(A_\theta)}.$$

Pf: Fix $\theta \in \{-1, 1\}^n$. Let $A = A_\theta$. By Lemma 3.2.5,

$$\tilde{P}(A) = \tilde{E}[1_A] = E[Z_n 1_A].$$

But $Z_n 1_A = \underbrace{Z_n(\theta)}_{\substack{\uparrow \\ \text{not random}}} 1_A$, so
↑ random

$$\tilde{P}(A) = E[Z_n(\theta) 1_A] = Z_n(\theta) E[1_A] = Z_n(\theta) P(A),$$

giving $Z_n(\theta) = \tilde{P}(A)/P(A)$. $\square$

For all $\theta \in \{-1, 1\}^n$,

$$Z_n(\theta) = \left(\frac{\tilde{q}}{q} \right)^n \left(\frac{\tilde{p}/\tilde{q}}{p/q} \right)^{\alpha(\theta)}$$

Pf: Follows from the fact that $P(A_\theta) = p^{\alpha(\theta)} q^{n-\alpha(\theta)}$
and $\tilde{P}(A_\theta) = \tilde{p}^{\alpha(\theta)} \tilde{q}^{n-\alpha(\theta)}$. $\square$

Lemma 3.2.6

Let $0 \leq m \leq n \leq N$. If Y depends only on

$\xi_1, \dots, \xi_n$, then

$$\tilde{E}_m[Y] = \frac{1}{Z_m} E_m[Z_n Y]$$

Remark: Lemma 3.2.5 say $\tilde{E}[Y] = E[Z_n Y]$.

That's equivalent to $\tilde{E}_0[Z_0 Y] = E_0[Z_n Y]$. Lemma

3.2.6 is equiv to $\tilde{E}_m[Z_m Y] = E_m[Z_n Y] \quad \forall m \leq n$.

Remark: Taking $n=N$, we get that for any r.v. Y ,

$$\tilde{E}_m[Y] = \frac{1}{Z_m} E_m[Z Y]$$

Pf: Note that $Z_m \tilde{E}_m[Y]$ depends only on $\xi_1, \dots, \xi_m$. Fix $\theta \in \{-1, 1\}^m$. Then

$$\begin{aligned} Z_m(\theta) \tilde{E}_m[Y](\theta) &= \frac{\tilde{P}(A_\theta)}{P(A_\theta)} \tilde{E}_m[Y](\theta) \\ &= \frac{1}{P(A_\theta)} \underbrace{\tilde{E}_m[Y](\theta)}_{\text{constant}} \tilde{E}[1_{A_\theta}] \\ &= \frac{1}{P(A_\theta)} \tilde{E}[\tilde{E}_m[Y](\theta) 1_{A_\theta}]. \end{aligned}$$

But $\tilde{E}_m[Y](\theta) 1_{A_\theta} = \tilde{E}_m[Y] 1_{A_\theta}$, so

$$\begin{aligned} Z_m(\theta) \tilde{E}_m[Y](\theta) &= \frac{1}{P(A_\theta)} \tilde{E}[\tilde{E}_m[Y] 1_{A_\theta}] \\ &= \frac{1}{P(A_\theta)} \tilde{E}[\tilde{E}_m[Y 1_{A_\theta}]] \\ &= \frac{1}{P(A_\theta)} \tilde{E}[Y 1_{A_\theta}] \end{aligned}$$

depends only on $\xi_1, \dots, \xi_n$

Lemma 3.2.5

$$\begin{aligned} &\stackrel{\curvearrowright}{=} \frac{1}{P(A_\theta)} E[Z_n Y 1_{A_\theta}] \\ &= \frac{1}{P(A_\theta)} E[E_m[Z_n Y 1_{A_\theta}]] \end{aligned}$$

(96)

$$= \frac{1}{P(A_\theta)} E[1_{A_\theta} \underbrace{E_m[Z_n Y]}_{\text{depends only on } \xi_1, \dots, \xi_m}].$$

depends only on $\xi_1, \dots, \xi_m$

But $1_{A_\theta} E_m[Z_n Y] = 1_{A_\theta} E_m[Z_n Y](\theta)$, so

$$Z_m(\theta) \tilde{E}_m[Y](\theta) = \frac{1}{P(A_\theta)} E[1_{A_\theta} \underbrace{E_m[Z_n Y](\theta)}_{\text{constant}}]$$

$$= \frac{1}{P(A_\theta)} E_m[Z_n Y](\theta) E[1_{A_\theta}]$$

$$= \frac{1}{\cancel{P(A_\theta)}} E_m[Z_n Y](\theta) \cancel{P(A_\theta)}$$

$$= E_m[Z_n Y](\theta).$$

Since $\theta \in \{-1, 1\}^m$ was arbitrary,

$Z_m \tilde{E}_m[Y] = E_m[Z_n Y]$, giving

$$\tilde{E}_m[Y] = \frac{1}{Z_m} E_m[Z_n Y]. \quad \square$$

Defn N3.2.4 The state price density process

is $\zeta = (\zeta_0, \dots, \zeta_N)$, where

↑
same notation as
state price density.
Must rely on
context

$$\zeta_n = \frac{Z_n}{(1+r)^n}$$

(So the state price density, ζ , satisfies

$$\zeta = \zeta_N, \text{ and } \zeta_0 = 1.)$$

Thm N3.2.5 (generalizes Thm N3.1.3)

If V_N is a deriv. secur., then $V_n = \frac{1}{\zeta_n} E_n[\zeta V_N]$.

Pf: By Lemma 3.2.6,

$$\begin{aligned} V_n &= \tilde{E}_n \left[\frac{V_N}{(1+r)^{N-n}} \right] = \frac{1}{Z_n} E_n \left[Z \cdot \frac{V_N}{(1+r)^{N-n}} \right] \\ &= \frac{(1+r)^n}{Z_n} E_n \left[\frac{Z}{(1+r)^N} V_N \right] = \frac{1}{\zeta_n} E_n[\zeta V_N]. \quad \square \end{aligned}$$

§ 3.3 Capital Asset Pricing Model

We shift focus here briefly.

Rather than pricing derivatives, we now want to maximize our wealth.

Lemma N3.3.1

Let X be the wealth process produced by (X_0, Δ) . Let $c \in \mathbb{R}$ and let X' be the wealth process produced by $(cX_0, c\Delta)$. Then $X'_n = cX_n$ for all n .

Pf idea: Induction on n , using

$$X_{n+1} = (1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}. \quad \square$$

Lemma N3.3.2

Let X be the wealth process produced by (X_0, Δ) and X' the wealth process produced by $(0, \Delta)$. Then $X_n = X'_n + (1+r)^n X_0$ for all n .

Pf idea: Same as above. $\square$

Problem N3.3.3 (not viable)

Given X_0 , find $\Delta = (\Delta_0, \Delta_1, \dots, \Delta_{N-1})$ that maximizes $E[X_N]$.

Thm N3.3.4

If $P = \tilde{P}$, then $E[X_N] = (1+r)^N X_0$ for all Δ .

Pf: Since $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -mart,

$$E[X_N] = (1+r)^N \tilde{E} \left[\frac{X_N}{(1+r)^N} \right] = (1+r)^N \tilde{E} \left[\frac{X_0}{(1+r)^0} \right]$$

$$= (1+r)^N X_0. \quad \square$$

Thm N3.3.5

Suppose $P \neq \tilde{P}$ and let $X_0 \in \mathbb{R}$ be given. For any $M > 0$, there exists Δ s.t. (X_0, Δ) yields a wealth process X satisfying $E[X_N] > M$.

Pf: Since $P \neq \tilde{P}$, we have $p \neq \tilde{p}$. Thus,

$pu + qd \neq 1+r$. Thus, we may choose $c \in \mathbb{R}$ s.t.

$$c((pu + qd)^N - (1+r)^N)S_0 + (1+r)^N X_0 > M.$$

$(pu + qd > 1+r \Rightarrow c > 0; pu + qd < 1+r \Rightarrow c < 0)$

Define $\Delta_n = c \quad \forall n$. Then (X_0, Δ) yields

$$X_N = (1+r)^N (X_0 - cS_0) + cS_N, \quad (\text{check})$$

so that

(100)

$$\begin{aligned} E[X_N] &= (1+r)^N (X_0 - cS_0) + c E[S_N] \\ &= (1+r)^N (X_0 - cS_0) + c (pu + qd)^N S_0 \\ &= c((pu + qd)^N - (1+r)^N) S_0 + (1+r)^N X_0 > M. \quad \square \end{aligned}$$

The last two thems show that Problem N3.3.3 is not viable. To make it viable, we maximize expected "utility", not wealth.

Defn N3.3.6

A function φ is (strictly) concave if $-\varphi$ is (strictly) convex.

Defn N3.3.7

A utility function is a real-valued, nondecreasing, concave function U with domain of the form (a, ∞) for some $a \in [-\infty, \infty)$. We will say $U(x) = -\infty$ for $x \leq a$.

Expl

natural logarithm.
in math, e is the
default base

$$U(x) = \begin{cases} \log x & \text{if } x > 0, \\ -\infty & \text{if } x \leq 0. \end{cases}$$

Jensen's inequ: $-U(E[X]) \leq E[-U(X)]$

$$\therefore \underbrace{U(E[X]) \geq E[U(X)]}_{\text{risk-aversion}}$$

Problem 3.3.1 (optimal investment)

Given X_0 , find $\Delta = (\Delta_0, \Delta_1, \dots, \Delta_{N-1})$ that maximizes $E[U(X_N)]$.

Expl 3.3.2

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, q=\frac{2}{3}$$

$$U(x) = \log x, X_0=4$$

Find Δ that maximizes $E[U(X_2)]$.

$$\Delta = (\Delta_0, \Delta_1), \quad \Delta_1 = \Delta_1(\xi_1).$$

Need to find 3 numbers: $\Delta_0, \Delta_1(1), \Delta_1(-1)$

Plan: Step 1: Find the maximum value, X_2 .

Step 2: Find the maximizer, Δ .

Step 1

$$X_{n+1} = (1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}$$

$$X_1 = \frac{5}{4}(4 - 4\Delta_0) + S_1 \Delta_0$$

$$= 5 - 5\Delta_0 + S_1 \Delta_0 = (S_1 - 5)\Delta_0 + 5$$

$$X_1(1) = 3\Delta_0 + 5, \quad X_1(-1) = -3\Delta_0 + 5$$

$$\begin{aligned} X_2 &= \frac{5}{4}(X_1 - \Delta_1 S_1) + \Delta_1 S_2 \\ &= \Delta_1(S_2 - \frac{5}{4}S_1) + \frac{5}{4}(\overbrace{(S_1 - 5)\Delta_0 + 5}^{X_1}) \\ &= \Delta_1(S_2 - \frac{5}{4}S_1) + \frac{5}{4}(S_1 - 5)\Delta_0 + \frac{25}{4} \end{aligned}$$

$$X_2(1,1) = 6\Delta_1(1) + \frac{15}{4}\Delta_0 + \frac{25}{4}$$

$$X_2(1,-1) = -6\Delta_1(1) + \frac{15}{4}\Delta_0 + \frac{25}{4}$$

$$X_2(-1,1) = \frac{3}{2}\Delta_1(-1) - \frac{15}{4}\Delta_0 + \frac{25}{4}$$

$$X_2(-1,-1) = -\frac{3}{2}\Delta_1(-1) - \frac{15}{4}\Delta_0 + \frac{25}{4}$$