

$$V_n = \frac{1}{1+r} \tilde{E}_n[V_{n+1}] = \frac{1}{1+r} \tilde{E}_n[v_{n+1}(Z_{n+1})]$$

this is the
key to the recursive
computation

$$= \frac{1}{1+r} \cdot \text{some function of } Z_n$$

Expl 1.3.2 revisited

$N = \text{any}$, $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $p = \text{any}$
was 3 before

$$M_n = \max\{S_0, \dots, S_n\}, \quad V_N = M_N - S_N$$

$$Z_n = (S_n, M_n), \quad Z \text{ is Markov}$$

$$V_N = v_N(Z_N), \quad v_N(s, m) = m - s$$

By Thm N2.5.3, $V_n = v_n(Z_n)$ for all n .

Find a recursive formula for $v_n(s, m)$ in terms of v_{n+1} .

$$\begin{aligned} V_n &= \frac{1}{1+r} \tilde{E}_n[V_{n+1}] = \frac{4}{5} \tilde{E}_n[v_{n+1}(S_{n+1}, M_{n+1})] \\ &= \frac{4}{5} \tilde{E}_n[v_{n+1}(Y_{n+1} S_n, M_n \vee (Y_{n+1} S_n))] \\ &= v_n(S_n, M_n), \quad \text{where} \end{aligned}$$

$$\begin{aligned}
V_n(s, m) &= \frac{4}{5} \tilde{E} [V_{n+1}(Y_{n+1}s, m v(Y_{n+1}s))] \\
&= \frac{4}{5} (\tilde{p} V_{n+1}(us, m v(us)) + \tilde{q} V_{n+1}(ds, m v(ds))) \\
&= \frac{2}{5} V_{n+1}(2s, m v(2s)) + \frac{2}{5} V_{n+1}(\frac{1}{2}s, m v(\frac{1}{2}s)).
\end{aligned}$$

We will only ever evaluate V_n when $s \leq m$. In that case, $m v(\frac{1}{2}s) = m$.

$$V_n(s, m) = \frac{2}{5} V_{n+1}(2s, m v(2s)) + \frac{2}{5} V_{n+1}(\frac{1}{2}s, m)$$

§ 2.6 Summary (2.4, 2.5 only)

$Z = (Z_0, \dots, Z_N)$ is a martingale under $\tilde{P}$ if it's adapted and

$$\tilde{E}_n [Z_{n+1}] = Z_n$$



$$\tilde{E}_n [Z_m] = Z_n \quad \forall m \geq n$$

In particular, $\tilde{E}_n [Z_N] = Z_n$.

- $\left\{ \frac{S_n}{(1+r)^n} \right\}$ is a mart. under $\tilde{P}$

- for any (X_0, Δ) , $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a mart. under $\tilde{\mathbb{P}}$
- given a deriv. sec. V_N , the proc. $V = (V_0, \dots, V_N)$ is defined recursively by $V_n = \frac{1}{1+r} \tilde{\mathbb{E}}_n[V_{n+1}]$.
- $\left\{ \frac{V_n}{(1+r)^n} \right\}$ is a mart. under $\tilde{\mathbb{P}}$

$$\therefore \tilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^N} \right] = \frac{V_n}{(1+r)^n}$$

$$\Updownarrow$$

$$V_n = \frac{1}{(1+r)^{N-n}} \tilde{\mathbb{E}}_n[V_N]$$

risk-neutral pricing formula

$Z = (Z_0, \dots, Z_N)$ is a Markov process under $\tilde{\mathbb{P}}$ if it's adapted and $\forall n, \forall f, \exists g$ s.t.

$$\tilde{\mathbb{E}}_n[f(Z_{n+1})] = g(Z_n) \quad \leftarrow \text{Markov property}$$

OR

$$\tilde{\mathbb{P}}(Z_{n+1} \in B | \xi_1, \dots, \xi_n) = \tilde{\mathbb{P}}(Z_{n+1} \in B | Z_n)$$

If $Z = (Z_0, \dots, Z_N)$ is Markov under $\tilde{P}$ and $V_N = v_N(Z_N)$, then $V_n = v_n(Z_n) \forall n$.

The functions v_n are computed recursively backwards using

$$\begin{aligned} V_n &= \frac{1}{1+r} \tilde{E}_n [V_{n+1}] \\ &= \frac{1}{1+r} \tilde{E}_n [v_{n+1}(Z_{n+1})] \\ &= \frac{1}{1+r} g(Z_n), \text{ for some } g \\ &\quad \uparrow \\ &\quad \text{Markov property} \end{aligned}$$

find g , and then $v_n(z) = \frac{1}{1+r} g(z)$.

HW: 2.3-2.7, 2.11(i)-(iii), 2.13

§3.1 Change of measure

Ω : finite sample space

$P, \tilde{P}$: prob measures on Ω

Assume $P(\omega) > 0$ and $\tilde{P}(\omega) > 0 \quad \forall \omega \in \Omega$.

$$Z(\omega) := \frac{\tilde{P}(\omega)}{P(\omega)}$$

↑
a random variable

called the Radon-Nikodym derivative
of $\tilde{P}$ with respect to P

Thm 3.1.1

(i) $P(Z > 0) = 1$

(iii) for any r.v. Y , $\tilde{E}[Y] = E[ZY]$

(iii') for any event A , $\tilde{P}(A) = E[Z1_A]$

(ii) $E[Z] = 1$

(iv) for any r.v. Y , $E[Y] = \tilde{E}[\frac{1}{Z}Y]$

(iv') for any event A , $P(A) = \tilde{E}[\frac{1}{Z}1_A]$

(ii') $\tilde{E}[\frac{1}{Z}] = 1$