

Expl

Let $M_n = \frac{S_n}{(1+r)^n}$. Show that $M = (M_0, \dots, M_N)$ is a martingale under $\tilde{P}$.

S_n is a function of $\xi_1, \dots, \xi_n$ and r is a constant, so M_n is a function of $\xi_1, \dots, \xi_n$. Thus,

M is adapted. ✓

$$\begin{aligned}\tilde{E}_n[M_{n+1}] &= \frac{1}{(1+r)^{n+1}} \tilde{E}_n[S_{n+1}] = \frac{1}{(1+r)^{n+1}} \tilde{E}_n[Y_{n+1} S_n] \\ &= \frac{1}{(1+r)^{n+1}} S_n \tilde{E}_n[Y_{n+1}] = \frac{1}{(1+r)^{n+1}} S_n (\tilde{p}u + \tilde{q}d) \\ &= \frac{1}{(1+r)^n} S_n = M_n \quad \checkmark\end{aligned}$$

Thm N2.4.1 If M is a mart. under P , then

$$E_n[M_m] = M_n \quad \forall m > n.$$

Pf: Equiv. to $E_n[M_{n+k}] = M_n \quad \forall k \geq 1$. True

by defn for $k=1$. If true for k , then

$$\begin{aligned}E_n[M_{n+k+1}] &= E_n[E_{n+1}[M_{n+1+k}]] \\ &= E_n[M_{n+1}] = M_n. \quad \square\end{aligned}$$

Thm N2.4.2 If M is a mart under P , then

$$E[M_n] = M_0 \quad \forall n.$$

Pf: $E[M_n] = E_0[M_n] = M_0. \quad \square$

NOTE: Analogous thms are true for sub & supermart.

Expl

$$N = \text{any}, S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, p = \frac{2}{3}$$

$$M_n = \frac{S_n}{(1+r)^n} = \left(\frac{4}{5}\right)^n S_n$$

Show that M is a submartingale under P .

S is adapted, so M is adapted $\checkmark$

$$\begin{aligned} E_n[M_{n+1}] &= \left(\frac{4}{5}\right)^{n+1} E_n[S_{n+1}] \\ &= \left(\frac{4}{5}\right)^{n+1} E_n[Y_{n+1} S_n] \\ &= \left(\frac{4}{5}\right)^{n+1} S_n E_n[Y_{n+1}] \\ &= \left(\frac{4}{5}\right)^{n+1} S_n \left(\underbrace{\frac{2}{3}}_p \underbrace{2}_u + \underbrace{\frac{1}{3}}_q \underbrace{\frac{1}{2}}_d \right) \\ &= \left(\frac{4}{5}\right)^{n+1} S_n \left(\frac{4}{3} + \frac{1}{6} \right) \frac{9}{6} = \frac{3}{2} \end{aligned}$$

$$= \frac{3}{2} \cdot \frac{4}{5} \cdot \left(\frac{4}{5}\right)^n S_n$$

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$$= \frac{6}{5} M_n > M_n. \checkmark$$

Thm 2.4.5

Let (X_0, Δ) be any trading strategy.

Let $X = (X_0, X_1, X_2, \dots, X_N)$ be the wealth

process determined by (X_0, Δ) . Define

$$M_n = \frac{X_n}{(1+r)^n}. \text{ Then } M = (M_0, \dots, M_N) \text{ is}$$

a martingale under $\tilde{P}$.

Pf: X_0 is a constant $\checkmark$

$$X_{n+1} = (1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}$$

X_1 depends on $\underbrace{X_0, \Delta_0, S_0}_{\text{constants}}, \underbrace{S_1}_{\text{depends on } \xi_1}$

So X_1 is a function of ξ_1 $\checkmark$

Suppose X_n is a function of $\xi_1, \dots, \xi_n$.

X_{n+1} depends on $\underbrace{X_n, \Delta_n, S_n}_{\text{depends on } \xi_1, \dots, \xi_n}, \underbrace{S_{n+1}}_{\text{depends on } \xi_1, \dots, \xi_{n+1}}$

(67)

So X_{n+1} depends on $\xi_1, \dots, \xi_{n+1}$.

By induction X_n depends on $\xi_1, \dots, \xi_n$

for all n . So X is adapted, and

hence, M is adapted. ✓

$$\begin{aligned}\tilde{E}_n[M_{n+1}] &= (1+r)^{-n-1} \tilde{E}_n[X_{n+1}] \\ &= (1+r)^{-n-1} \tilde{E}_n[(1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}] \\ &= (1+r)^{-n-1} \left((1+r) \underbrace{\tilde{E}_n[X_n - \Delta_n S_n]}_{\text{function of } \xi_1, \dots, \xi_n} + \tilde{E}_n[\underbrace{\Delta_n S_{n+1}}_{= ?}] \right)\end{aligned}$$

$$= (1+r)^{-n-1} \left((1+r)(X_n - \Delta_n S_n) + \Delta_n \underbrace{\tilde{E}_n[S_{n+1}]}_{= ?} \right)$$

$$\begin{aligned}\tilde{E}_n[S_{n+1}] &= (\tilde{p}u + \tilde{q}d) S_n \\ &= (1+r) S_n\end{aligned}$$

$$\begin{aligned}\tilde{E}_n[M_{n+1}] &= (1+r)^{-n-1} \left((1+r)(X_n - \Delta_n S_n) + (1+r) \Delta_n S_n \right) \\ &= (1+r)^{-n} X_n = M_n. \quad \square\end{aligned}$$

By Thm N2.4.2, a martingale has constant expectation. So:

$$X_0 = \tilde{\mathbb{E}}\left[\frac{X_1}{(1+r)}\right] = \tilde{\mathbb{E}}\left[\frac{X_2}{(1+r)^2}\right] = \dots = \tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right]$$

Two probability measures, Q and $\tilde{Q}$, are equivalent if, for all events A , we have $Q(A) = 0$ iff $\tilde{Q}(A) = 0$.

Thm N2.4.3 In the binomial model, P and $\tilde{P}$ are equivalent.

Pf: Recall that $p, q > 0$ by the assumptions of our model. Also, $\tilde{p} = \frac{1+r-d}{u-r}$. Since

$d < 1+r < u$, we have $\tilde{p} \in (0, 1)$, so that

$\tilde{p}, \tilde{q} > 0$ also. Thus, $P(\omega) = p^{\alpha(\omega)} q^{N-\alpha(\omega)} > 0$

$\forall \omega \in \Omega$ and $\tilde{P}(\omega) = \tilde{p}^{\alpha(\omega)} \tilde{q}^{N-\alpha(\omega)} > 0 \quad \forall \omega \in \Omega$.

Hence, $P(A) = 0$ iff $A = \emptyset$ iff $\tilde{P}(A) = 0$. $\square$

We call $\tilde{P}$ the "risk-neutral prob. measure." (69)

It's also called the equivalent martingale measure, since it's equivalent to P and makes the discounted stock price into a martingale.

Thm N1.2.2 (again) There is no arbitrage in the multiperiod binomial model.

Pf: Suppose that (X_0, Δ) is an arbitrage strategy. Let $X = (X_0, X_1, \dots, X_N)$ be the wealth process determined by (X_0, Δ) . Since $P(X_N > 0) > 0$ and X_N is discrete, there exists $x > 0$ s.t.

$P(X_N = x) > 0$. Since P and $\tilde{P}$ are equiv., this means $\tilde{P}(X_N = x) > 0$. Since $P(X_N < 0) = 0$, this means $X_N \geq 0$, and therefore,

$$X_N \geq X_N 1_{\{X_N = x\}} = x 1_{\{X_N = x\}}.$$

Thus, since $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -martingale, (70)

we have

$$\begin{aligned} 0 = X_0 &= \frac{X_0}{(1+r)^0} = \tilde{\mathbb{E}} \left[\frac{X_N}{(1+r)^N} \right] \\ &= \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[X_N] \geq \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[x 1_{\{X_N=x\}}] \\ &= \frac{x}{(1+r)^N} \tilde{\mathbb{P}}(X_N=x) > 0, \end{aligned}$$

a contradiction. $\square$

Proof of Thm N1.2.4 (the strategy (X_0, Δ) that

replicates V_N is unique) Let $(X_0, \Delta), (X'_0, \Delta')$

both replicate V_N . Let $X = (X_0, \dots, X_N)$ and

$X' = (X'_0, \dots, X'_N)$ be their wealth processes, so that

$X_N = X'_N = V_N$. Let $n \in \{1, \dots, N\}$. Since $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is

a $\tilde{\mathbb{P}}$ -mart.,

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n \left[\frac{X_N}{(1+r)^N} \right] = \tilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^N} \right],$$

which gives $X_n = \frac{1}{(1+r)^{N-n}} \tilde{\mathbb{E}}_n[V_N]$. Similarly,

(71)

$$X'_n = \frac{1}{(1+r)^{N-n}} \tilde{E}_n[V_N]. \text{ So } X_n = X'_n \quad \forall n. \text{ In}$$

particular, $X_0 = X'_0$. Solving

$$X_{n+1} = (1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}, \text{ and}$$

$$X'_{n+1} = (1+r)(X'_n - \Delta'_n S_n) + \Delta'_n S_{n+1}$$

for Δ_n and Δ'_n , and using $X_n = X'_n \quad \forall n$, we

get $\Delta_n = \Delta'_n \quad \forall n. \quad \square$

Thm N2.4.4 Let V_N be a derivative security and $V = (V_0, V_1, \dots, V_N)$ its price process. Then

$\left\{ \frac{V_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -martingale. Consequently,

$$V_n = \frac{1}{(1+r)^{N-n}} \tilde{E}_n[V_N] \quad \forall n.$$

In particular, $V_0 = \frac{1}{(1+r)^N} \tilde{E}[V_N]$.

Pf: Let (X_0, Δ) replicate V_N . Then $X_n = V_n \quad \forall n$.

Thus, $\left\{ \frac{V_n}{(1+r)^n} \right\} = \left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -martingale. $\square$

Cash flow

A cash flow is like a derivative security, but instead of paying at time N , it pays something at every time n . It is modeled by an adapted process $C = (C_0, C_1, \dots, C_N)$, where C_n represents the payout at time n . (C_n could be negative.)

Derivative sec. is special case: $(0, 0, \dots, 0, V_N)$

Imagine you can buy and sell them on Reddit. Some of the cash flows on Reddit are officially approved by Portfolio Inc. for use in their Portfolio+ app.

What you see after initial transfer:

MM BROKER CASH FLOW

REGISTER CASH FLOW
AS SELLER

ID #

BUYER'S BANK INFO

OK

What you see next, and then again every morning

MM BROKER CASH FLOW

CURRENT VALUE OF
PORTFOLIO: \$...

CASH FLOW OWED
TODAY: \$...

DAY CASH FLOW
FROM MM

value before
paying today's
cash flow

if today is Day n,
this is C_n

goes to recipient's
bank acct

Can't do anything until paying cash flow.

This is a new model. MM balance changes because of cash flow.

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X_n := value of Portfolio before paying cash flow on Day n .

in mm at end of Day n : $X_n - C_n - \Delta_n S_n$

of shares at end of Day n : Δ_n

$$X_{n+1} = (1+r)(X_n - C_n - \Delta_n S_n) + \Delta_n S_{n+1} \quad (*)$$

Define

$$V_n := \tilde{E}_n \left[\sum_{k=n}^N \frac{C_k}{(1+r)^{k-n}} \right]$$

• net present value
at time n of
 $C_n, C_{n+1}, \dots, C_N$

• $V_N = C_N$

and

$$\Delta_n = \frac{V_{n+1}(\xi_1, \dots, \xi_n, 1) - V_{n+1}(\xi_1, \dots, \xi_n, -1)}{S_{n+1}(\xi_1, \dots, \xi_n, 1) - S_{n+1}(\xi_1, \dots, \xi_n, -1)}.$$

V_n is the (no-arbitrage) price of the cash flow
at time n .

Thm 2.4.8 If $X_0 = V_0$, then the trading strategy (X_0, Δ) determines a wealth process according to (*) that satisfies $X_n = V_n \quad \forall n$.

Pf: See book. $\square$

At the end, you have $X_N = C_N$, make final payment, and you break even.

§2.5 Markov processes

In this class, a K-dimensional stochastic process is a finite sequence of $\mathbb{R}^K$ -valued random vectors, $Z = (Z_0, \dots, Z_N)$.

Expl

$$M_n = \max_{0 \leq j \leq n} S_j$$

$(M, S) = \{(M_0, S_0), (M_1, S_1), \dots, (M_N, S_N)\}$ is a 2-dimensional stochastic process.

Defn 2.5.1/2.5.5

Let $Z = (Z_0, \dots, Z_N)$ be a k -dim stoch. proc. that is adapted. Then Z is a Markov process if for all nonnegative integers n and all function f , there exists a function g such

that

Markov under $\tilde{P}$. Use $\tilde{E}_n$ for Markov under $\tilde{P}$

$$\tilde{E}_n[f(Z_{n+1})] = g(Z_n)$$

can depend on n and f

called the Markov property

Thm N2.5.1 An adapted process Z is a Markov process iff for all n and A ,

$$P(Z_{n+1} \in A | \xi_1, \dots, \xi_n) = P(Z_{n+1} \in A | Z_n)$$

Pf: Beyond the scope of the course.

This is what is usually called the "Markov property"

Thm N2.5.2 If Z is a Markov process, f is a func., and m, n are nonneg. ints w/ $n \leq m$, then $\exists$ a func g s.t.

$$E_n[f(Z_m)] = g(Z_n)$$

can depend on n, m, f

Pf sketch:

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$$E_n[f(Z_{n+2})] = E_n[E_{n+1}[f(Z_{n+2})]]$$

$h(Z_{n+1})$

depends on f and $n+1$
(so on f and n)

$$= E_n[h(Z_{n+1})] = g(Z_n)$$

depends on h and n
(so on f and n)

Induction... $\square$

Expl 2.5.2

Show that $S = (\underline{S_0, S_1, \dots, S_N})$ is Markov under P .

S is adapted $\checkmark$

$$E_n[f(S_{n+1})] = E_n[f(Y_{n+1} S_n)]$$

conditioning on
 $\xi_1, \dots, \xi_n$

indep. of
 $\xi_1, \dots, \xi_n$

function of
 $\xi_1, \dots, \xi_n$

$$E[f(Y_{n+1} s)] =: g(s) \quad , \quad E_n[f(S_{n+1})] = g(S_n). \checkmark$$

Not needed, but

$$g(s) = p f(us) + q f(ds)$$

S is also Markov under $\tilde{P}$.

Expl 2.5.2

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$$N = \text{any}, S_0 = 4, u = 2, d = \frac{1}{2}, r = \text{any}, p = \frac{2}{3}$$

$$M_n = \max_{0 \leq j \leq n} S_j. \text{ Show that } M = (M_0, \dots, M_N)$$

is not Markov under P .

Suppose it is. Let $n=2$, $f(x)=x$.

Then there is some g such that

$$E_2[M_3] = E_n[f(M_{n+1})] = g(M_n) = g(M_2).$$

Notice that

$$g(M_2(-1, 1)) = g(4) = g(M_2(-1, -1)).$$

Thus,

$$E_2[M_3](-1, 1) = E_2[M_3](-1, -1).$$

However,

$$E_2[M_3](-1, 1) = E[M_3 \mid \xi_1 = -1, \xi_2 = 1]$$

$$= E[M_3(-1, 1, \xi_2)]$$

$$= p M_3(-1, 1, 1) + q M_3(-1, 1, -1)$$

$$= \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 4 = \frac{20}{3},$$

and

$$\begin{aligned} E_2[M_3](-1, -1) &= p M_3(-1, -1, 1) + q M_3(-1, -1, -1) \\ &= \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 4 = \frac{12}{3} = 4, \end{aligned}$$

a contradiction. $\square$ (Also not Markov under $\tilde{P}$.)

Expl 2.5.6

Show that $\{(S_n, M_n)\}_{n=0}^N$ is Markov under P .

adapted $\checkmark$

$$\begin{aligned} E_n[f(S_{n+1}, M_{n+1})] &= E_n[f(Y_{n+1} S_n, M_n \vee S_{n+1})] \\ &= E_n[f(Y_{n+1} S_n, M_n \vee (Y_{n+1} S_n))] \end{aligned}$$

$$E[f(Y_{n+1} s, m \vee (Y_{n+1} s))] =: g(s, m)$$

$$E_n[f(S_{n+1}, M_{n+1})] = g(S_n, M_n) \checkmark$$

Not needed, but

$$g(s, m) = p f(us, m \vee (us)) + q f(us, m \vee (ds)).$$

Also Markov under $\tilde{P}$.

Thm N2.5.3

Let V_N be a derivative security. Suppose there is a proc. Z that is Markov under $\tilde{P}$ and a func. v_N s.t. $V_N = v_N(Z_N)$. Then for all n , there is a function v_n s.t. $V_n = v_n(Z_n)$.

Pf: By the pricing formula and our hypothesis,

$$V_n = \frac{1}{(1+r)^{N-n}} \tilde{E}_n[V_N] = \frac{1}{(1+r)^{N-n}} \tilde{E}_n[v_N(Z_N)].$$

By the Markov prop., $V_n = \frac{1}{(1+r)^{N-n}} g(Z_n)$,

for some g that depends on v_N , n , and N .

Then $v_n(z) = \frac{1}{(1+r)^{N-n}} g(z)$. $\square$

We compute the v_n 's recursively backward, starting with v_N .

$\left\{ \frac{V_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -martingale

$$\frac{V_n}{(1+r)^n} = \tilde{E}_n \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right]$$

$$V_n = \frac{1}{1+r} \tilde{E}_n[V_{n+1}] = \frac{1}{1+r} \tilde{E}_n[v_{n+1}(Z_{n+1})]$$

this is the
key to the recursive
computation

$$= \frac{1}{1+r} \cdot \text{some function of } Z_n$$

Expl 1.3.2 revisited

$N = \text{any}$, $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $p = \text{any}$
was 3 before

$$M_n = \max\{S_0, \dots, S_n\}, \quad V_N = M_N - S_N$$

$$Z_n = (S_n, M_n), \quad Z \text{ is Markov}$$

$$V_N = v_N(Z_N), \quad v_N(s, m) = m - s$$

By Thm N2.S.3, $V_n = v_n(Z_n)$ for all n .

Find a recursive formula for $v_n(s, m)$ in terms of v_{n+1} .

$$\begin{aligned} V_n &= \frac{1}{1+r} \tilde{E}_n[V_{n+1}] = \frac{4}{5} \tilde{E}_n[v_{n+1}(S_{n+1}, M_{n+1})] \\ &= \frac{4}{5} \tilde{E}_n[v_{n+1}(Y_{n+1} S_n, M_n \vee (Y_{n+1} S_n))] \\ &= v_n(S_n, M_n), \quad \text{where} \end{aligned}$$