

Expl 1.2.4

52

$$N=3, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\text{any}$$

$$M_3 = \max_{0 \leq n \leq 3} S_n, \quad V_3 = M_3 - S_3$$

(a) Find the price of this derivative at time 0.

(b) Consider an investor that is replicating

(or hedging) this derivative. How many

shares do they buy on Day 0? (Give a

negative number if they should sell instead.)

Describe their portfolio at the end of Day 0.

(c) Suppose the stock goes up on Day 1

(meaning $S_1 > S_0$).

How many shares should they buy on Day 1?

Describe their portfolio at the end of Day 1.

(d) Suppose the stock then goes down on Day 2.

How many shares should they buy on Day 2?

Describe their portfolio at the end of Day 2.

(e) Finally, suppose goes down again on Day 3. (53)

Verify that investor's wealth on Day 3 matches the payout of the derivative.

(a)	ξ_1, ξ_2, ξ_3	M_3	S_3	V_3
	1, 1, 1	32	32	0
	1, 1, -1	16	8	8
	1, -1, 1	8	8	0
	1, -1, -1	8	2	6
	-1, 1, 1	8	8	0
	-1, 1, -1	4	2	2
	-1, -1, 1	4	2	2
	-1, -1, -1	4	$\frac{1}{2}$	$\frac{7}{2}$

$\tilde{p} = \tilde{q} = \frac{1}{2}$ (same as last example)

$$V_2(\xi_1, \xi_2) = \frac{1}{1+r} (\tilde{p} V_3(\xi_1, \xi_2, 1) + \tilde{q} V_3(\xi_1, \xi_2, -1))$$

$$V_2(1, 1) = \frac{2}{5} \overset{0}{V_3(1, 1, 1)} + \frac{2}{5} \overset{8}{V_3(1, 1, -1)} = \frac{16}{5}$$

$$V_2(1, -1) = \frac{2}{5} \overset{0}{V_3(1, -1, 1)} + \frac{2}{5} \overset{6}{V_3(1, -1, -1)} = \frac{12}{5}$$

$$V_2(-1, 1) = \frac{2}{5} \overset{0}{V_3(-1, 1, 1)} + \frac{2}{5} \overset{2}{V_3(-1, 1, -1)} = \frac{4}{5}$$

$$V_2(-1, -1) = \frac{2}{5} \overset{2}{V_3(-1, -1, 1)} + \frac{2}{5} \overset{\frac{7}{2}}{V_3(-1, -1, -1)} = \frac{11}{5}$$

$$V_1(\xi_1) = \frac{1}{1+r} (\tilde{p} V_2(\xi_1, 1) + \tilde{q} V_2(\xi_1, -1))$$

$$V_1(1) = \frac{2}{5} \cancel{V_2(1, 1)} + \frac{2}{5} \cancel{V_2(1, -1)}$$

$\frac{16}{5}$ $\frac{12}{5}$

$$= \frac{32}{25} + \frac{24}{25} = \frac{56}{25}$$

$$V_1(-1) = \frac{2}{5} \cancel{V_2(-1, 1)} + \frac{2}{5} \cancel{V_2(-1, -1)}$$

$\frac{4}{5}$ $\frac{11}{5}$

$$= \frac{8}{25} + \frac{22}{25} = \frac{30}{25} = \frac{6}{5}$$

$$V_0 = \frac{1}{1+r} (\tilde{p} V_1(1) + \tilde{q} V_1(-1))$$

$$= \frac{2}{5} \cdot \frac{56}{25} + \frac{2}{5} \cdot \frac{30}{25} = \frac{112 + 60}{125}$$

$$= \frac{172}{125} = \boxed{\$1.376}$$

$$(b) \quad \Delta_0 = \frac{V_1(1) - V_1(-1)}{S_1(1) - S_1(-1)}$$

$$= \frac{\frac{56}{25} - \frac{30}{25}}{8 - 2} = \frac{26}{25} \cdot \frac{1}{6} = \boxed{\frac{13}{75} \text{ shares}}$$

$$\text{mm} : \frac{172}{125} - \frac{13}{75} (4) = \frac{172(3) - 52(5)}{375} = \frac{256}{375}$$

$$\text{mm} : \frac{256}{375} \text{ dollars} = \$0.682\bar{6}$$

$$\text{shares} : \frac{13}{75} = 0.17\bar{3}$$

(55)

$$(c) \quad \xi_1 = 1, \quad S_1 = 8$$

$$\Delta_1(1) = \frac{V_2(1,1) - V_2(1,-1)}{S_2(1,1) - S_2(1,-1)}$$

$$= \frac{\frac{16}{5} - \frac{12}{5}}{16 - 4} = \frac{4}{5} \cdot \frac{1}{12} = \frac{1}{15}$$

$$\text{buy } \frac{1}{15} - \frac{13}{75} = \frac{5-13}{75} = \boxed{-\frac{8}{75} \text{ shares}}$$

$$\text{Start of Day 1:} \quad \text{mm: } \frac{256}{375} \left(1 + \frac{1}{4}\right) = \frac{64}{75}$$

$$\text{shares: } \frac{13}{75}$$

$$\text{End of Day 1:} \quad \text{mm: } \frac{64}{75} + \frac{8}{75}(8) = \frac{128}{75}$$

$$\boxed{\begin{array}{l} \text{mm: } \frac{128}{75} \text{ dollars} = \$1.70\overline{6} \\ \text{shares: } \frac{1}{15} = 0.0\overline{6} \end{array}}$$

$$(d) \quad \xi_1 = 1, \quad \xi_2 = -1, \quad S_2 = 4$$

$$\Delta_2(1,-1) = \frac{V_3(1,-1,1) - V_3(1,-1,-1)}{S_3(1,-1,1) - S_3(1,-1,-1)}$$

$$= \frac{0 - 6}{8 - 2} = \frac{-6}{6} = -1$$

$$\text{buy } -1 - \frac{1}{15} = \boxed{-\frac{16}{15} \text{ shares}}$$

Start of Day 2: $mm: \frac{128}{75} \cdot \frac{5}{4} = \frac{32}{15}$
 shares: $\frac{1}{15}$

End of Day 2: $mm: \frac{32}{15} + \frac{16}{15}(4) = \frac{96}{15}$

$mm: \frac{96}{15} \text{ dollars} = \$6.40$

shares: -1

(e) $\xi_1 = 1, \xi_2 = -1, \xi_3 = -1, S_3 = 2$

$mm: \frac{96}{15} \cdot \frac{5}{4} = \frac{24}{3} = 8 \text{ dollars}$

shares: -1

wealth: $8 + (-1)(2) = 6 \text{ dollars}$

$V_3(1, -1, -1) = 6 \checkmark$

HW: 1.3, 1.6

§1.3 Computational considerations

(57)

Expl 1.3.1

$$N=3, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\text{any}$$

$$\tilde{p} = \tilde{q} = \frac{1}{2} \text{ as before}$$

$$V_3 = (5 - S_3)^+ \leftarrow \text{Euro. put option w/ } K=5$$

Find V_0 , price of Euro. put at time 0.

$$\boxed{v_3(s) := (5 - s)^+} \Rightarrow V_3 = v_3(S_3)$$

Will show that $\forall n, \exists$ a func v_n s.t. $V_n = v_n(S_n)$

Backwards induction:

Assume $V_{n+1} = v_{n+1}(S_{n+1})$ for some func v_{n+1} .

$$V_n = \frac{1}{1+r} (\tilde{p} V_{n+1}(\xi_1, \dots, \xi_n, 1) + \tilde{q} V_{n+1}(\xi_1, \dots, \xi_n, -1))$$

$$V_{n+1}(\xi_1, \dots, \xi_n, 1) = v_{n+1}(S_{n+1}(\xi_1, \dots, \xi_n, 1))$$

$$= v_{n+1}(2S_n(\xi_n, \dots, \xi_1))$$

$$= v_{n+1}(2S_n)$$

$$\begin{aligned}
 V_{n+1}(\xi_1, \dots, \xi_n, 1) &= V_{n+1}(S_{n+1}(\xi_1, \dots, \xi_n, 1)) \\
 &= V_{n+1}\left(\frac{1}{2} S_n(\xi_1, \dots, \xi_n)\right) \\
 &= V_{n+1}\left(\frac{1}{2} S_n\right)
 \end{aligned}$$

58

$$V_n = \frac{1}{1+r} (\tilde{p} V_{n+1}(2S_n) + \tilde{q} V_{n+1}(\frac{1}{2} S_n))$$

define $V_n(s) = \frac{1}{1+r} (\tilde{p} V_{n+1}(2s) + \tilde{q} V_{n+1}(\frac{1}{2} s))$

Then $V_n = \underline{V_n(S_n)}$.

$$V_0 = v_0(S_0) = v_0(4).$$

In this example, $r = \frac{1}{4}$, $\tilde{p} = \tilde{q} = \frac{1}{2}$, so

$$V_n(s) = \frac{2}{5} V_{n+1}(2s) + \frac{2}{5} V_{n+1}(\frac{1}{2} s)$$

$$V_0(4) = \frac{2}{5} V_1(8) + \frac{2}{5} V_1(2)$$

$$= \frac{2}{5} \left(\frac{2}{5} V_2(16) + \frac{2}{5} V_2(4) \right) + \frac{2}{5} \left(\frac{2}{5} V_2(4) + \frac{2}{5} V_2(1) \right)$$

$$= \frac{4}{25} V_2(16) + \frac{8}{25} V_2(4) + \frac{4}{25} V_2(1)$$

$$= \frac{8}{125} V_3(32) + \frac{8}{125} V_3(8)$$

$$+ \frac{16}{125} V_3(4) + \frac{16}{125} V_3(2)$$

$$+ \frac{8}{125} V_3(1) + \frac{8}{125} V_3(\frac{1}{2})$$

$$V_3(s) = (5-s)^+$$

(59)

$$= \frac{8}{125} V_3(32) + \frac{24}{125} V_3(8) + \frac{24}{125} V_3(2) + \frac{8}{125} V_3\left(\frac{1}{2}\right)$$

$$= \frac{72}{125} + \frac{36}{125} = \boxed{\frac{108}{125}} = \$0.864$$

As a bonus, this formula:

$$\Delta_n(\xi_1, \dots, \xi_n) = \frac{V_{n+1}(\xi_1, \dots, \xi_n, 1) - V_{n+1}(\xi_1, \dots, \xi_n, -1)}{S_{n+1}(\xi_1, \dots, \xi_n, 1) - S_{n+1}(\xi_1, \dots, \xi_n, -1)}$$

becomes $\boxed{\Delta_n = \delta_n(S_n)}$ where

$$\delta_n(s) = \frac{V_{n+1}(2s) - V_{n+1}\left(\frac{1}{2}s\right)}{2s - \frac{1}{2}s}$$

$$\boxed{\delta_n(s) = \frac{2}{3s} (V_{n+1}(2s) - V_{n+1}\left(\frac{1}{2}s\right))}$$

Expl 1.3.2

$$N=3, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\text{any}$$

$$\tilde{p} = \tilde{q} = \frac{1}{2}, \text{ as before}$$

$$M_n = \max \{S_0, S_1, \dots, S_n\}$$

$$V_3 = M_3 - S_3$$

Find V_0 .

$$\boxed{V_3(s, m) = m - s} \Rightarrow V_3 = v_3(S_3, M_3)$$

Will show that $\forall n, \exists$ a func v_n s.t. $V_n = v_n(S_n, M_n)$

Backwards induction:

Assume $V_{n+1} = v_{n+1}(S_{n+1}, M_{n+1})$ for some func v_{n+1} .

$$V_n = \frac{1}{1+r} (\tilde{p} V_{n+1}(\xi_1, \dots, \xi_n, 1) + \tilde{q} V_{n+1}(\xi_1, \dots, \xi_n, -1))$$

$$S_{n+1}(\xi_1, \dots, \xi_n, -1) = \frac{1}{2} S_n, \quad S_{n+1}(\xi_1, \dots, \xi_n, 1) = 2S_n$$

$$M_{n+1}(\xi_1, \dots, \xi_n, -1) = M_n, \quad M_{n+1}(\xi_1, \dots, \xi_n, 1) = ?$$

$$\text{In general, } M_{n+1} = \begin{cases} M_n & \text{if } S_{n+1} \leq M_n, \\ S_{n+1} & \text{if } S_{n+1} > M_n. \end{cases}$$

$$\underline{\text{OR}} \quad M_{n+1} = M_n \vee S_{n+1}$$

So...

$$\begin{aligned} M_{n+1}(\xi_1, \dots, \xi_n, 1) &= M_n \vee S_{n+1}(\xi_1, \dots, \xi_n, 1) \\ &= M_n \vee (2S_n) \end{aligned}$$

$$V_n = \frac{1}{1+r} (\tilde{p} v_{n+1}(2S_n, M_n v(2S_n))$$

(61)

$$+ \tilde{q} v_{n+1}(\frac{1}{2}S_n, M_n))$$

define

$$v_n(s, m) = \frac{1}{1+r} (\tilde{p} v_{n+1}(2s, m v(2s)) + \tilde{q} v_{n+1}(\frac{1}{2}s, m))$$

Then $V_n = \underline{v_n(S_n, M_n)}$

$$V_0 = v_0(4, 4)$$

In this example,

$$v_n = \frac{2}{5} v_{n+1}(2s, m v(2s)) + \frac{2}{5} v_{n+1}(\frac{1}{2}s, m)$$

$$v_0(4, 4) = \frac{2}{5} v_1(8, 8) + \frac{2}{5} v_1(2, 4)$$

$$= \frac{4}{25} v_2(16, 16) + \frac{4}{25} v_2(4, 8) + \frac{4}{25} v_2(4, 4) + \frac{4}{25} v_2(1, 4)$$

$$= \frac{8}{125} v_3(\overset{0}{\cancel{32}}, \overset{0}{\cancel{32}}) + \frac{8}{125} v_3(\overset{8}{\cancel{8}}, \overset{8}{\cancel{16}}) + \frac{8}{125} v_3(\overset{0}{\cancel{8}}, \overset{0}{\cancel{8}}) + \frac{8}{125} v_3(\overset{6}{\cancel{2}}, \overset{6}{\cancel{8}})$$

$$+ \frac{8}{125} v_3(\overset{0}{\cancel{8}}, \overset{0}{\cancel{8}}) + \frac{8}{125} v_3(\overset{2}{\cancel{2}}, \overset{2}{\cancel{4}}) + \frac{8}{125} v_3(\overset{2}{\cancel{2}}, \overset{2}{\cancel{4}}) + \frac{8}{125} v_3(\overset{\frac{7}{2}}{\cancel{\frac{1}{2}}}, \overset{\frac{7}{2}}{\cancel{4}})$$

$$v_3(s, m) = m - s$$

$$= \frac{1}{125} (64 + 48 + 16 + 16 + 28)$$

$$= \boxed{\frac{172}{125}} = \$1.376$$

As a bonus, this formula:

$$\Delta_n(\xi_1, \dots, \xi_n) = \frac{V_{n+1}(\xi_1, \dots, \xi_n, 1) - V_{n+1}(\xi_1, \dots, \xi_n, -1)}{S_{n+1}(\xi_1, \dots, \xi_n, 1) - S_{n+1}(\xi_1, \dots, \xi_n, -1)}$$

becomes $\boxed{\Delta_n = \delta_n(S_n, M_n)}$ where

$$\delta_n(s, m) = \frac{V_{n+1}(2s, mv(2s)) - V_{n+1}(\frac{1}{2}s, m)}{2s - \frac{1}{2}s}$$

$$\boxed{\delta_n(s, m) = \frac{2}{3s} (V_{n+1}(2s, mv(2s)) - V_{n+1}(\frac{1}{2}s, m))}$$

Not every derivative security can be analyzed this way. We'll look more at this technique in § 2.5.

HW: Exer. 1.8

§ 1.4 Summary (includes 2.1-2.3)

63a

Stock model

Coin toss space: $\Omega = \Omega_N = \{-1, 1\}^N$

$$\xi_n(\omega) = \omega_n, \quad \alpha(\omega) = |\xi_n: \omega_n = 1|$$

$$p \in (0, 1), \quad q = 1 - p, \quad P = P^p, \quad P^p(\omega) = p^{\alpha(\omega)} q^{N - \alpha(\omega)}$$

$$S_0 > 0, \quad 0 < d < u$$

$$1 \leq n \leq N: \quad Y_n = \begin{cases} u & \text{if } \xi_n = 1, \\ d & \text{if } \xi_n = -1 \end{cases}$$

$$S_n = Y_n S_{n-1}$$

Every random var. on coin toss sp. is a function of the coin flips: $X = X(\xi_1, \dots, \xi_N)$

Conditional expectation

$$E_n[X] = E[X | \xi_1, \dots, \xi_n], \quad E_0[X] = E[X]$$

$E_n[X]$ is a function of only $\xi_1, \dots, \xi_n$:

$$E_n[X] = E_n[X](\xi_1, \dots, \xi_n)$$

$$E_n[X](x_1, \dots, x_n) = E[X | \xi_1 = x_1, \dots, \xi_n = x_n]$$

Properties

- $E_n[aX + bY] = aE_n[X] + bE_n[Y]$
- $X = X(\xi_1, \dots, \xi_n) \Rightarrow E_n[XY] = X E_n[Y]$
- $m < n \Rightarrow E_m[E_n[X]] = E_n[E_m[X]] = E_m[X]$
- Y and $(\xi_1, \dots, \xi_n)$ indep $\Rightarrow E_n[Y] = E[Y]$
- $X = X(\xi_1, \dots, \xi_n)$ and Y and $(\xi_1, \dots, \xi_n)$ indep
 $\Rightarrow E_n[f(X, Y)] = g(X)$, where $g(x) = E[f(x, Y)]$

Portfolio

- (i) stock (see above)
- (ii) money market
- (iii) investor

The money market is modeled by a daily interest rate $r > -1$ satisfy the no-arbitrage assumption, $d < 1+r < u$. The investor's choices are modeled by a trading strategy (X_0, Δ) , where $X_0 \in \mathbb{R}$ and $\Delta = (\Delta_0, \Delta_1, \dots, \Delta_{N-1})$ is an adapted process. A derivative security is modeled by a r.v. V_N .

(63c)

The risk-neutral probability measure is

$\tilde{P} = P^{\tilde{P}}$, where $\tilde{P} = \frac{1+r-d}{u-d}$. The risk-neutral probabilities are the unique probabilities satisfying $\tilde{p}u + \tilde{q}d = 1+r$, or equivalently,

$$\tilde{E}_n[S_{n+1}] = (1+r)S_n.$$

Under $\tilde{P}$, the stock (on average) has the same growth rate as the money market.

We define $V_n = \frac{1}{1+r} \tilde{E}_n[V_{n+1}]$ and call V_n the no-arbitrage price of the derivative at time n .

If we choose $X_0 = V_0$ and

$$\Delta_n = \frac{V_{n+1}(\xi_1, \dots, \xi_n, 1) - V_{n+1}(\xi_1, \dots, \xi_n, -1)}{S_{n+1}(\xi_1, \dots, \xi_n, 1) - S_{n+1}(\xi_1, \dots, \xi_n, -1)},$$

then the strategy (X_0, Δ) will replicate V_N , meaning $X_N = V_N$. In fact, $X_n = V_n \forall n$.

§ 2.4 Martingales

63d

In this class, a stochastic process is a finite sequence of random variables, $Z = (Z_0, \dots, Z_N)$.

(In general, a stochastic process is any indexed collection of r.v.s)

A stochastic process Z is adapted if Z_0 is a constant (i.e. not random) and Z_n is a function of $\xi_1, \dots, \xi_n$ for all $n > 0$.

Suppose $M = (M_0, \dots, M_N)$ is adapted. If

$E_n[M_{n+1}] = M_n \quad \forall n$, then M is a

martingale (under P). If $E_n[M_{n+1}] \geq M_n \quad \forall n$,

then M is a submartingale (under P). If

$E_n[M_{n+1}] \leq M_n \quad \forall n$, then M is a

supermartingale (under P).

Expl

(64)

Let $M_n = \frac{S_n}{(1+r)^n}$. Show that $M = (M_0, \dots, M_N)$ is a martingale under $\tilde{P}$.

S_n is a function of $\xi_1, \dots, \xi_n$ and r is a constant, so M_n is a function of $\xi_1, \dots, \xi_n$. Thus,

M is adapted. ✓

$$\begin{aligned}\tilde{E}_n[M_{n+1}] &= \frac{1}{(1+r)^{n+1}} \tilde{E}_n[S_{n+1}] = \frac{1}{(1+r)^{n+1}} \tilde{E}_n[Y_{n+1} S_n] \\ &= \frac{1}{(1+r)^{n+1}} S_n \tilde{E}_n[Y_{n+1}] = \frac{1}{(1+r)^{n+1}} S_n (\tilde{p}u + \tilde{q}d) \\ &= \frac{1}{(1+r)^n} S_n = M_n \quad \checkmark\end{aligned}$$

Thm N2.4.1 If M is a mart. under P , then

$$E_n[M_m] = M_n \quad \forall m > n.$$

Pf: Equiv. to $E_n[M_{n+k}] = M_n \quad \forall k \geq 1$. True

by defn for $k=1$. If true for k , then

$$\begin{aligned}E_n[M_{n+k+1}] &= E_n[E_{n+1}[M_{n+1+k}]] \\ &= E_n[M_{n+1}] = M_n. \quad \square\end{aligned}$$