

Put it together:

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$$E_1^P[E_2^P[S_3]](1) = \frac{2}{3} \cdot 24 + \frac{1}{3} \cdot 6$$
$$= 16 + 2 = \boxed{18}$$

Ch. 1 The binomial model

"Portfolio" app

MM **BROKER**

WELCOME

☐ - day plan

OK

"money market"

MM **BROKER**

BALANCE: \$ 0

ONE-TIME TRANSFER

AMOUNT: \$

BANK INFO:

OK

can be negative

negative allowed, 0 also allowed

required

MM **BROKER**

TRADE ABC, INC.

SHARES OWNED: 0

PRICE/SHARE: \$ 4

QUANTITY:

BUY

can be negative, need not be integers

- MM earns interest (charges same interest when negative)
- Share purchases use money from MM

3-day plan expl:

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Day 0	day you start trade all you want
sleep	stock price changes
Day 1	trade all you want
sleep	stock price changes
Day 2	trade all you want
sleep	stock price changes
Wake up on Day 3	plan has ended no trading allowed broker tab is resolved to 0 MM is transferred to bank

§1.1 One-period binomial model

This is the 1-day plan.

Stock price is modeled w/ binomial model:

$$N=1, S_0 > 0, 0 < d < u, p \in (0,1)$$

$$\Omega = \Omega_1 = \{-1, 1\} \text{ (sample space)}$$

$$P = P^P \text{ (prob. measure)}$$

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$$P(1) = p, \quad P(-1) = q$$

modeling the money market

r : interest rate

$\$a$ on Day 0 becomes $\$(1+r)a$ on Day 1
(true whether $a > 0$, $a = 0$, or $a < 0$)

assume $r > -1$.

modeling the investor's choices:

X_0 : amount of initial transfer
↖ initial wealth

Δ_0 : # of shares bought on Day 0

X_0, Δ_0 can be any real numbers

(X_0, Δ_0) called a trading strategy

X_1 = value of everything in Portfolio app on Day 1
this is what is transferred to bank on Day 1
called wealth at time 1

$X_1 = ?$ How to calculate?

X_0 : initial wealth

$\Delta_0 S_0$: price of shares

$X_0 - \Delta_0 S_0$: amount in mm at end of Day 0

$$X_1 = \underbrace{(1+r)(X_0 - \Delta_0 S_0)}_{\text{amount in mm on Day 1}} + \underbrace{\Delta_0 S_1}_{\text{value of shares on Day 1}}$$

OR

$$X_1 = \underbrace{X_0}_{\substack{\uparrow \\ \text{initial} \\ \text{wealth}}} + \underbrace{r(X_0 - \Delta_0 S_0)}_{\substack{\text{interest} \\ \text{from mm}}} + \underbrace{\Delta_0 (S_1 - S_0)}_{\substack{\text{profit/loss} \\ \text{from trading}}}$$

Arbitrage

Ordinary language:

"arbitrage" = "money for nothing"

Mathematically:

We will use the word "arbitrage" in multiple mathematical definitions, all having something to do with getting money for nothing.

Defn N1.1.1 An arbitrage (strategy) is a trading strategy (X_0, Δ_0) such that

(i) $X_0 = 0$,

(ii) $P(X_1 < 0) = 0$, and

(iii) $P(X_1 > 0) > 0$.

Expl

$N=1$, $S_0=4$, $u=3$, $d=1.5$, $r=\frac{1}{4}$, $p=\text{any}$

In this case, stock goes up even on a "down" day (just not as much as on an up day).

In fact, even on a "down" day, stock does better than mm.

In this expl, $(X_0, \Delta_0) = (0, 1)$ is an arbitrage strategy.

INFORMAL EXPLANATION

No initial transfer, immediately buy 1 share.

Portfolio at end of Day 0: mm: -4
shares: 1

Portfolio at end of Day 1: MM: -5
 shares: 1
 interest

Worst case, stock is worth $4(1.5) = 6$

Your 1 share is sold, leaving at least \$1 in mm, which is transferred to bank.
 (You started with nothing)

FORMAL EXPLANATION

$$X_0 = 0 \checkmark$$

$$\begin{aligned} X_1 &= (1 + \frac{1}{4}) (X_0 - \Delta_0 S_0) + \Delta_0 S_1 \\ &= \frac{5}{4} (-4) + \gamma_1 S_0^4 \\ &= -5 + 4\gamma_1 \end{aligned}$$

$$= \begin{cases} -5 + 4\mu^3 & \text{if } \xi_1 = 1 \\ -5 + 4d^{\frac{3}{2}} & \text{if } \xi_1 = -1 \end{cases}$$

$$= \begin{cases} 7 & \text{if } \xi_1 = 1 \\ 1 & \text{if } \xi_1 = -1 \end{cases}$$

$$P(X_1 < 0) = 0 \checkmark$$

$$P(X_1 > 0) = 1 > 0 \checkmark$$

Using this argument, we can prove that in general,

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if $(1+r) \leq d < u$, then $(X_0, \Delta_0) = (0, 1)$ is an arbitrage

Expl

$N=1, S_0=4, u=\frac{5}{4}, d=\frac{3}{4}, r=\frac{1}{4}, p=\text{any}$

In this case, even on a good day, the mm does at least as well as the stock. (There's no reason to buy the stock.)

Here, $(X_0, \Delta_0) = (0, -1)$ is an arbitrage.

INFORMAL EXPLANATION

No initial transfer, immediately sell 1 share.

Portfolio at end of Day 0: mm: 4
shares: -1

Portfolio at end of Day 1: mm: 6 (interest)
shares: -1

Plan is over, shares are resolved.

Broker uses mm to buy 1 share to settle your account.

Worst case, that share costs
 $(\frac{5}{4})4 = 5$ dollars, leaving you at least
 $\$1$ in the mm.

FORMAL EXPLANATION

$$X_0 = 0 \checkmark$$

$$X_1 = (1 + \frac{1}{4}) (X_0 - \Delta_0 S_0) + \Delta_0 S_1$$

$$= \frac{5}{4} (4) - \gamma_1 S_0^4 = 5 - 4\gamma_1$$

$$= \begin{cases} 5 - 4u^{\frac{5}{4}} & \text{if } \xi_1 = 1 \\ 5 - 4d^{\frac{3}{4}} & \text{if } \xi_1 = -1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } \xi_1 = 1 \\ 1 & \text{if } \xi_1 = -1 \end{cases}$$

$$P(X_1 < 0) = 0 \checkmark$$

$$P(X_1 > 0) = P(\xi_1 = -1) = q > 0 \checkmark$$

Using this argument, we can prove that in general,

if $d < u \leq 1+r$, then $(X_0, \Delta_0) = (0, -1)$ is an arbitrage

Exercise 1.1 in the book has you show that if $d < 1+r < u$, then there is no arbitrage.

HW: Exer. 1.1

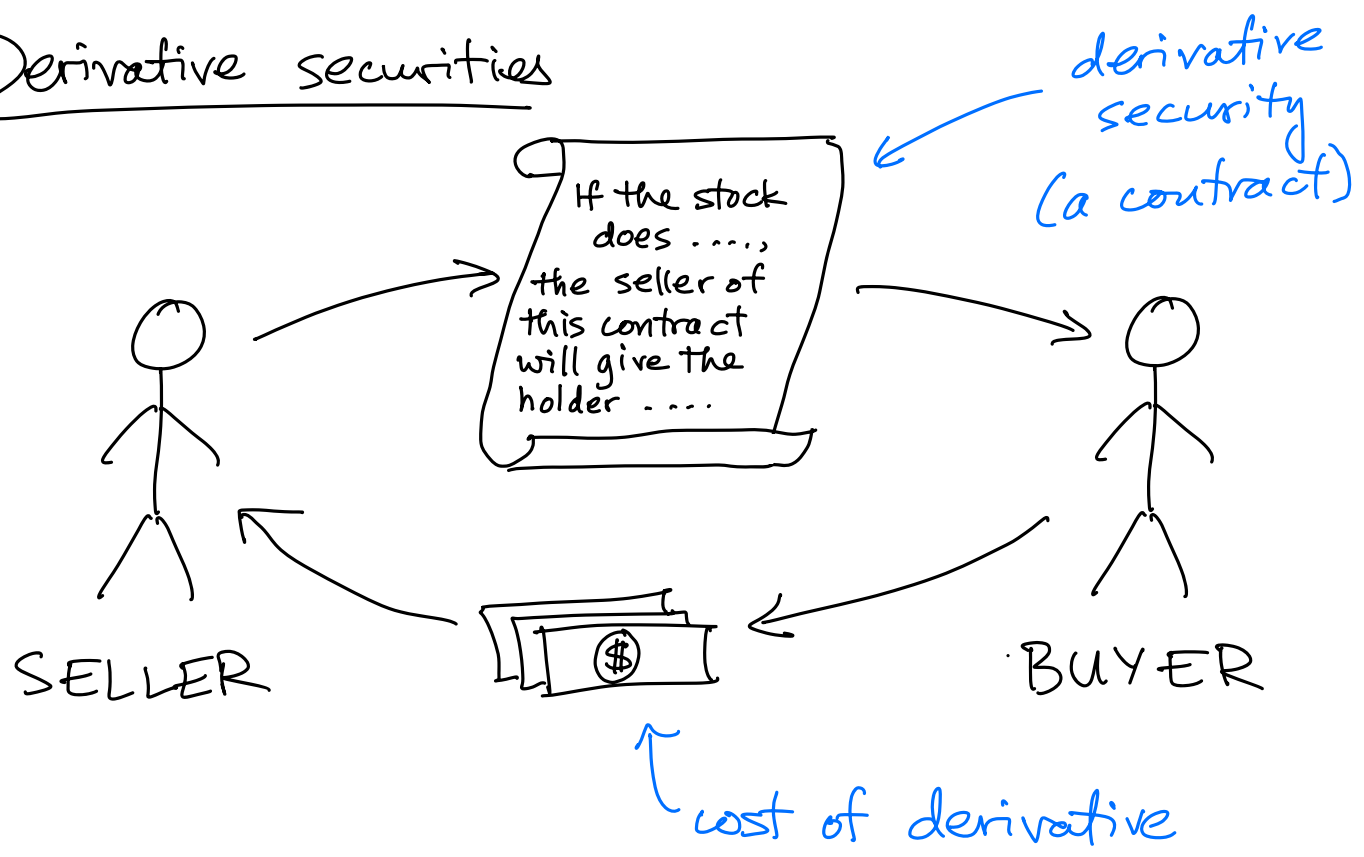
Putting it together:

Thm N1.1.2 In the 1-period binomial model, there is no arbitrage iff $d < 1+r < u$.

From now on, we will assume that

$$d < 1+r < u$$

Derivative securities



Imagine you can buy and sell these contracts on Reddit, e.g.

(It's external to the Portfolio app in our story.)

Every derivative security can be modeled

by a r.v. $V_1 = V_1(\xi_1)$

the dollar amount the seller must give the buyer (depends on ξ_1)

Conversely, every r.v. describes some derivative security.

Defn N1.1.3 A derivative security

is a r.v. $V_1 = V_1(\xi_1)$

Expl

European call option w/strike price K