

Ch. 5 Random Walk

5.1 Introduction

$$\Omega = \Omega_\infty = \{-1, 1\}^\infty$$

$$= \{\omega = (\omega_1, \omega_2, \dots) : \text{each } \omega_n = -1 \text{ or } 1\}$$

$$\xi_n(\omega) = \omega_n$$

Choose $p \in (0, 1)$. Let $q := 1 - p$

$\mathbb{P}$: a prob. meas that makes

$$\xi_1, \xi_2, \dots \text{ iid}$$

with $\mathbb{P}(\xi_n = 1) = p$, $\mathbb{P}(\xi_n = -1) = q$

$$M_0 = 0$$

$$M_n = \sum_{j=1}^n \xi_j$$

$$M = \{M_n\}_{n=0}^\infty$$

↑
stoch. proc., adapted (each M_n depends only on $\xi_1, \dots, \xi_n$)

If $p = \frac{1}{2}$, then M is a symmetric random walk

If $p \neq \frac{1}{2}$, then M is an asymmetric random walk

- M is a Markov proc.
- $p = \frac{1}{2} \Rightarrow M$ is a mart.

5.2 First passage times

For $m \in \mathbb{Z}$,

$$\tau_m := \min \{ n : M_n = m \}$$

$= 1^{\text{st}} \text{ time walk hits } m$

$$\tau_0 = 0$$

$\{ \tau_m = \infty \}$: "The walk never hits m ."

Lemma 5.2.1

Let $p = \frac{1}{2}$ and $\sigma \in \mathbb{R}$. Define

$$S_n = e^{\sigma M_n} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n$$

Then $\{ S_n \}_{n=0}^{\infty}$ is a mart.

Pf: It was HW. $\square$

"almost surely" (a.s.) means

"with probability one"

Then S.S.2 Let $p = \frac{1}{2}$ and let $m \in \mathbb{Z}$ be arbitrary. Then $\tau_m < \infty$ a.s. (i.e. $P(\tau_m < \infty) = 1$)

Pf:

Case 1 ($m=0$)

$$X_0 = 0 \text{ a.s.} \Rightarrow \tau_0 = 0 \text{ a.s.} \Rightarrow \tau_0 < \infty \text{ a.s.}$$

Case 2 ($m > 0$)

Fix $\sigma > 0$. Let S_n be as in Lemma 5.2.1,

so that $\{S_n\}$ is a mart. By optional

sampling, $\{S_n \wedge \tau_m\}$ is a mart. $\therefore$

$$1 = E[S_0] = E[S_n \wedge \tau_m]$$

$$= E \left[\underbrace{\frac{e^{\sigma M_n \wedge \tau_m}}{\cosh n \tau_m \sigma}}_{\rightarrow ? \text{ as } n \rightarrow \infty} \right]$$

$$\tau_m(\omega) = \infty \Rightarrow n \wedge \tau_m(\omega) = n \quad \forall n$$

$$\cosh^n \sigma \xrightarrow{n \rightarrow \infty} \infty$$

↑
($\sigma > 0 \Rightarrow \cosh \sigma > 1$)

$M_n < m \quad \forall n$ ($M_n(\omega)$ starts at 0 and never hits $m > 0$)

$$\therefore e^{\sigma M_n} \in (0, e^{\sigma m}) \quad \forall n$$

$$\therefore \frac{e^{\sigma M_n}}{\cosh^n \sigma} \rightarrow 0$$

In other words,

$$1_{\{\tau_m = \infty\}} \frac{e^{\sigma M_n \wedge \tau_m}}{\cosh^{n \wedge \tau_m} \sigma} \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

$\tau_m(\omega) < \infty \Rightarrow n \wedge \tau_m(\omega) = \tau_m(\omega)$ for large enough n .

$$\text{and } M_{\tau_m} = m$$

$\therefore$

$$1_{\{\tau_m < \infty\}} \frac{e^{\sigma M_n \wedge \tau_m}}{\cosh^{n \wedge \tau_m} \sigma} \xrightarrow{n \rightarrow \infty} 1_{\{\tau_m < \infty\}} \frac{e^{\sigma m}}{\cosh^{\tau_m} \sigma}$$

Combine:

$$\frac{e^{\sigma M_n \wedge \tau_m}}{\cosh^{n \wedge \tau_m} \sigma} =$$

$$1_{\{\tau_m = \infty\}} \frac{e^{\sigma M_{n\wedge \tau_m}}}{\cosh^{n\wedge \tau_m} \sigma} + 1_{\{\tau_m < \infty\}} \frac{e^{\sigma M_{n\wedge \tau_m}}}{\cosh^{n\wedge \tau_m} \sigma}$$

$$\xrightarrow{n \rightarrow \infty} 1_{\{\tau_m < \infty\}} \frac{e^{\sigma m}}{\cosh^{\tau_m} \sigma}$$

$$\therefore 1 = \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} E \left[\frac{e^{\sigma M_{n\wedge \tau_m}}}{\cosh^{n\wedge \tau_m} \sigma} \right]$$

$$= E \left[\lim_{n \rightarrow \infty} \frac{e^{\sigma M_{n\wedge \tau_m}}}{\cosh^{n\wedge \tau_m} \sigma} \right]$$

high-powered theorem from Vol. 2,
"dominated convergence thm" or DCT

$$= E \left[1_{\{\tau_m < \infty\}} \frac{e^{\sigma m}}{\cosh^{\tau_m} \sigma} \right]$$

$$\therefore 1 = \lim_{\sigma \rightarrow 0} 1 = \lim_{\sigma \rightarrow 0} E \left[1_{\{\tau_m < \infty\}} \frac{e^{\sigma m}}{\cosh^{\tau_m} \sigma} \right]$$

$$= E \left[\lim_{\sigma \rightarrow 0} 1_{\{\tau_m < \infty\}} \frac{e^{\sigma m}}{\cosh^{\tau_m} \sigma} \right]$$

DCT again

$$= E[1_{\{\tau_m < \infty\}}] = P(\tau_m < \infty).$$

Case 3 ($m < 0$)

symmetry (since $p = \frac{1}{2}$) . $\square$

Thm 5.2.3

If $p = \frac{1}{2}$, then $\forall m \in \mathbb{Z}$ and $\forall r \in (0, 1)$,

$$\underbrace{E[r^{\tau_m}]}_{\text{moment generating function of } \tau_m} = \left(\frac{1 - \sqrt{1-r^2}}{r} \right)^{|m|}$$

Pf:

Case 1 ($m=0$)

$\tau_0 = 0$, both sides equal 1. $\checkmark$

Case 2 ($m > 0$)

$$1 = \overset{\text{from previous proof}}{E \left[1_{\{\tau_m < \infty\}} \frac{e^{\sigma m}}{\cosh \tau_m \sigma} \right]}$$

$$\overset{\text{since } \tau_m < \infty \text{ a.s.}}{=} E \left[\frac{e^{\sigma m}}{\cosh \tau_m \sigma} \right]$$

$$\Rightarrow E\left[\left(\frac{1}{\cosh \sigma}\right)^{\tau_m}\right] = e^{-\sigma m}$$

Take $\sigma = \cosh^{-1}\left(\frac{1}{r}\right) > 1$

Then

$$E[r^{\tau_m}] = \underbrace{\left(e^{-\cosh^{-1}\left(\frac{1}{r}\right)}\right)^m}_{(*)}$$

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$$

$$\begin{aligned} \therefore (*) &= \frac{1}{\frac{1}{r} + \sqrt{\frac{1}{r^2} - 1}} \cdot \frac{r}{r} = \frac{r}{1 + \sqrt{1 - r^2}} \cdot \frac{1 - \sqrt{1 - r^2}}{1 - \sqrt{1 - r^2}} \\ &= \frac{r(1 - \sqrt{1 - r^2})}{1 - (1 - r^2)} = \frac{1 - \sqrt{1 - r^2}}{r} \quad \checkmark \end{aligned}$$

Case 3 ($m < 0$) symmetry. $\square$

Cor 5.2.4

Let $p = \frac{1}{2}$ and $m \in \mathbb{Z}$.

If $m \neq 0$, then $E[\tau_m] = \infty$.

Pf: Case 1 ($m > 0$)

Fix $r \in (0, 1)$. Then

$$\tau_m \geq \tau_1 \geq \tau_1 r^{\tau_1 - 1}$$

↑ must hit 1 on the way to m

$$\therefore E[\tau_m] \geq E[\tau_1 r^{\tau_1 - 1}]$$

$$= E\left[\frac{d}{dr}(r^{\tau_1})\right]$$

$$= \frac{d}{dr} E[r^{\tau_1}]$$

↑ high-powered thm. from Vol. 2,
derived from DCT

$$= \frac{d}{dr} \left(\frac{1 - \sqrt{1-r^2}}{r^2} \right)$$

$$= \dots = \frac{1 - \sqrt{1-r^2}}{r^2 \sqrt{1-r^2}} \geq \left(\frac{1}{\sqrt{1-r^2}} - 1 \right)$$

True for all $r \in (0, 1)$. Let $r \uparrow 1$. ✓

Case 2 ($m < 0$) symmetry. $\square$

Thm $\forall j \in \mathbb{N}, P(\tau_1 = 2j-1) = \frac{(2j-2)!}{j!(j-1)!} p^j q^{j-1}$

Pf: Fix $j \in \mathbb{N}$. Let $S = \{-1, 1\}^{2j-1}$.

Let

$$A = \{\theta \in S : \theta_1 + \theta_2 + \dots + \theta_{2j-1} = 1 \text{ and}$$

$$\forall k < 2j-1, \theta_1 + \dots + \theta_k < 1\}$$

Let $\xi = (\xi_1, \dots, \xi_{2j-1})$. Then

$$P(\tau_1 = 2j-1) = P(\xi \in A) = \sum_{\theta \in A} P(\xi = \theta)$$

Given $\theta = (\underbrace{\theta_1, \dots, \theta_{2j-1}}_{\theta}) \in A$

$$(\# \text{ of } 1\text{'s}) - (\# \text{ of } -1\text{'s}) = 1$$

$$\therefore \# \text{ of } 1\text{'s} = j, \quad \# \text{ of } -1\text{'s} = j-1$$

$$\therefore P(\xi = \theta) = p^j q^{j-1}$$

$$\therefore P(\tau_1 = 2j-1) = \sum_{\theta \in A} p^j q^{j-1} = \underbrace{|A|}_{\substack{\text{\# of elements in } A}} p^j q^{j-1}$$

$$|A| = ?$$

Suppose for the moment that $p = \frac{1}{2}$.

$$\text{Let } f(r) = E[r^{\tau_1}] = \frac{1 - \sqrt{1-r^2}}{r} = (*)$$

Since $\tau_1 \in \{1, 3, 5, 7, \dots\}$, we have

$$E[r^{\tau_1}] = \sum_{j=1}^{\infty} r^{2j-1} P(\tau_1 = 2j-1)$$

see book
for details

Compare this to the power series for (*) to get

$$P(\tau_1 = 2j-1) = \frac{(2j-2)!}{j!(j-1)!} \left(\frac{1}{2}\right)^{2j-1}$$

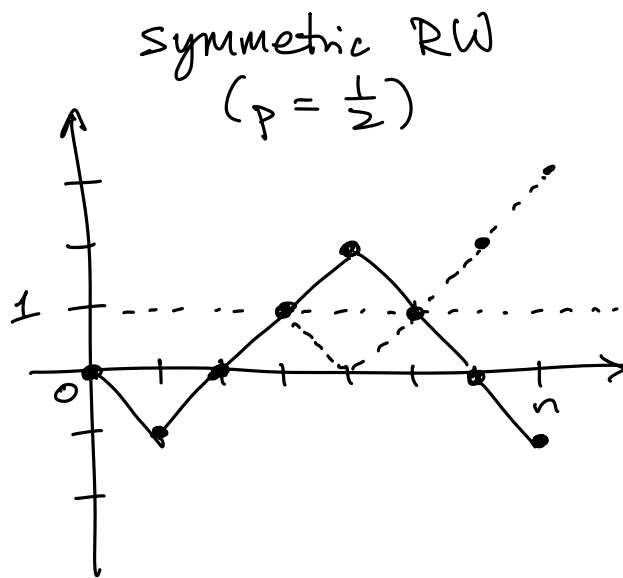
$$= \frac{(2j-2)!}{j!(j-1)!} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{j-1} = |A| p^j q^{j-1}$$

$$\therefore |A| = \frac{(2j-2)!}{j!(j-1)!}$$

So for any p , we have

$$P(\tau_1 = 2j-1) = \frac{(2j-2)!}{j!(j-1)!} p^j q^{j-1}. \quad \square$$

5.3 Reflection principle



$$\begin{aligned} & \{\tau_1 \leq n\} \cap \{M_n > 1\} \\ &= \{\tau_1 \leq n\} \cap \{M_n < 1\} \\ & \quad \uparrow \\ & \text{reflection principle} \end{aligned}$$

$$\begin{aligned} P(\tau_1 \leq n) &= P(\tau_1 \leq n, M_n < 1) + P(\tau_1 \leq n, M_n > 1) \\ &\quad + P(\tau_1 \leq n, M_n = 1) \end{aligned}$$

$$= P(\tau_1 \leq n, M_n = 1) + 2P(\tau_1 \leq n, M_n > 1)$$

$$M_n = 1 \Rightarrow \tau_1 \leq n \quad \text{and} \quad M_n > 1 \Rightarrow \tau_1 \leq n$$

$$\therefore \{M_n = 1\} \subset \{\tau_1 \leq n\}$$

$$\{M_n > 1\} \subset \{\tau_1 \leq n\}$$

$$\begin{aligned} \{\tau_1 \leq n\} \cap \{M_n = 1\} \\ = \{M_n = 1\} \end{aligned}$$

$$\begin{aligned} \{\tau_1 \leq n\} \cap \{M_n > 1\} \\ = \{M_n > 1\} \end{aligned}$$

$$\therefore \boxed{P(\tau_1 \leq n) = P(M_n = 1) + 2P(M_n > 1)}$$