

Thm 4.2.2 shows that the time 0 price cannot be greater than V_0 . If the price were $x > V_0$, the seller could sell the derivative for x , pocket $x - V_0$, and use $X_0 = V_0$ to replicate with nonnegative consumption. If the buyer exercises at time τ , the seller will have $V_\tau \geq G_\tau^+$.

The seller will pay the buyer G_τ , keep $V_\tau - G_\tau \geq 0$, and keep any consumption prior to τ .

But what if the price were $x < V_0$?

How can the buyer create an arbitrage from this? Based on what we saw with European derivatives we expect that they would capitalize on the cheap price by using

the strategy, $(-X_0, -\Delta, -C)$. The

initial wealth $-X_0$ means that at the start, Portfolio Inc. puts X_0 dollars into their bank account. They could use x of it to buy the American derivative, and pocket the rest. But now they have the American derivative. They have to decide when to exercise it. So they pick a stopping time τ . At time τ , they will get G_τ and have wealth $-V_\tau$, for a net of $G_\tau - V_\tau$. Thus, they must pick τ so that $G_\tau - V_\tau \geq 0$. Is this possible?

Even if it is, there's another problem.

Their consumption process is $-C$. Since

$C_n \geq 0 \forall n$, this means their consumption is always either negative or zero. They don't

want negative consumption, since that would mean depositing more money. Maybe this could also be solved by picking the right τ . After all, they can quit trading after they exercise the American derivative.

So they'll be all right as long as

$$C_n 1_{\xi_n < \tau} = 0 \quad \forall n. \quad \left(\text{Don't need } C_\tau = 0 \text{ since you exercise the option before doing the day's consumption.} \right)$$

In summary, if the price is less than V_0 , then the buyer can form an arbitrage, provided they can find a stopping time τ such that $V_\tau \leq G_\tau$ and $C_n 1_{\xi_n < \tau} = 0 \quad \forall n$.

Thm 4.4.5 + (optimal exercise)

Let $G = (G_0, \dots, G_N)$ be the intrinsic value process of an American derivative security, and $V = (V_0, \dots, V_N)$ its price process. Let (X_0, Δ, C) be the trading

strategy that replicates V . Define

$$\tau^* = \min \{n : V_n = G_n\}.$$

Then

τ^* is a maximizer in the defn of V_0

$$(i) \quad V_0 = \mathbb{E} \left[1_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*} \right]$$

$$(ii) \quad V_{\tau^*} = G_{\tau^*} \quad \text{when} \quad \tau^* < \infty \quad \text{and}$$

$$V_N = 0 \quad \text{when} \quad \tau^* = \infty$$

$\tau^* = \infty$ means you never exercise; in that case, your final wealth is $-V_N = 0$

$$(iii) \quad C_n 1_{\{n < \tau^*\}} = 0 \quad \forall n.$$

Pf: We first prove (ii). By Thm 4.4.2,

$$V_n \geq G_n^+ \geq G_n. \quad \text{Thus, either } V_n > G_n \text{ or } V_n = G_n.$$

Therefore, $V_n > G_n$ when $n < \tau^*$ and $V_{\tau^*} = G_{\tau^*}$

when $\tau^* < \infty$. When $\tau^* = \infty$, we have $N < \tau^*$.

Therefore $V_N > G_N$ when $\tau^* = \infty$. But Lemma N4.4.1

implies $V_N = G_N^+ = G_N \vee 0$. If $V_N > G_N$, then

we must have $V_N = 0$.

We next prove (iii). By Thm 4.4.3,

$$V_n = G_n \vee \left(\frac{1}{1+r} \tilde{E}_n[V_{n+1}] \right)$$

Thus, $n < \tau^* \Rightarrow V_n > G_n \Rightarrow V_n = \frac{1}{1+r} \tilde{E}_n[V_{n+1}]$.

By Thm. N4.4.2, $C_n = V_n - \frac{1}{1+r} \tilde{E}_n[V_{n+1}] = 0$.

Hence $C_n 1_{\tau^* > n} = 0$, and this prove (iii).

Lastly, we prove (i). Let $M_n = \frac{1}{(1+r)^{n \wedge \tau^*}} V_{n \wedge \tau^*}$.

We will show that $\{M_n\}$ is a $\tilde{P}$ -mart.

$$\tilde{E}_n[M_{n+1}] = \tilde{E}_n[1_{\{\tau^* > n\}} M_{n+1} + 1_{\{\tau^* \leq n\}} M_{n+1}]$$

$$= \underbrace{\tilde{E}_n[1_{\{\tau^* > n\}} M_{n+1}]}_{(*)} + \underbrace{\tilde{E}_n[1_{\{\tau^* \leq n\}} M_{n+1}]}_{(**)}$$

$$(*) = \tilde{E}_n[1_{\{\tau^* > n\}} \frac{1}{(1+r)^{(n+1) \wedge \tau^*}} V_{(n+1) \wedge \tau^*}]$$

$$= \tilde{E}_n[1_{\{\tau^* > n\}} \frac{1}{(1+r)^{n+1}} V_{n+1}]$$

$$= \tilde{E}_n[\tilde{E}_n[1_{\{\tau^* > n\}} \frac{1}{(1+r)^{n+1}} V_{n+1}]]$$

$$1_{\{\tau^* > n\}} = 1 - 1_{\{\tau^* \leq n\}} = 1 - \sum_{k=0}^n \underbrace{1_{\{\tau^* = k\}}}_{\text{depends only on } \xi_1, \dots, \xi_k, k \leq n}$$

So $1_{\{\tau^* > n\}}$ depends only on $\xi_1, \dots, \xi_n$

$$\therefore (*) = \tilde{E}_n \left[1_{\{\tau^* > n\}} \frac{1}{(1+r)^{n+1}} \tilde{E}_n[V_{n+1}] \right]$$

Recall that $\tau^* > n \Rightarrow V_n > G_n \Rightarrow$

$$V_n = G_n \vee \left(\frac{1}{1+r} \tilde{E}_n[V_{n+1}] \right) > G_n$$

$$\Rightarrow V_n = \frac{1}{1+r} \tilde{E}_n[V_{n+1}]$$

$$\begin{aligned} \therefore (*) &= \tilde{E}_n \left[1_{\{\tau^* > n\}} \frac{1}{(1+r)^n} V_n \right] \\ &= \tilde{E}_n \left[1_{\{\tau^* > n\}} \frac{1}{(1+r)^{n+\tau^*}} V_{n+\tau^*} \right] \\ &= \tilde{E}_n \left[1_{\{\tau^* > n\}} M_n \right] = 1_{\{\tau^* > n\}} M_n. \end{aligned}$$

$$\begin{aligned} (***) &= \tilde{E}_n \left[1_{\{\tau^* \leq n\}} \frac{1}{(1+r)^{(n+1)+\tau^*}} V_{(n+1)+\tau^*} \right] \\ &= \tilde{E}_n \left[1_{\{\tau^* \leq n\}} \frac{1}{(1+r)^{n+\tau^*}} V_{n+\tau^*} \right] \\ &= \tilde{E}_n \left[1_{\{\tau^* \leq n\}} M_n \right] = 1_{\{\tau^* \leq n\}} M_n. \end{aligned}$$

$$\begin{aligned}
\therefore \tilde{E}_n[M_{n+1}] &= (*) + (***) \\
&= 1_{\{\tau^* > n\}} M_n + 1_{\{\tau^* \leq n\}} M_n \\
&= M_n. \checkmark
\end{aligned}$$

$\therefore \{M_n\}$ is a $\tilde{P}$ -mart.

We therefore have

$$\begin{aligned}
V_0 = M_0 &= \tilde{E}[M_N] \\
&= \tilde{E}\left[\frac{1}{(1+r)^{N \wedge \tau^*}} V_{N \wedge \tau^*}\right] \\
&\stackrel{\text{by (ii)}}{=} \tilde{E}\left[1_{\{\tau^* < \infty\}} \frac{1}{(1+r)^{N \wedge \tau^*}} V_{N \wedge \tau^*}\right] \\
&= \tilde{E}\left[1_{\{\tau^* < \infty\}} \frac{1}{(1+r)^{\tau^*}} V_{\tau^*}\right] \\
&= \tilde{E}\left[1_{\{\tau^* < \infty\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}\right] \\
&= \tilde{E}\left[1_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}\right]. \square
\end{aligned}$$

4.5 American call option

Thm 4.5.1 Suppose $G = (G_0, \dots, G_N)$ satisfies $G_n = g(S_n)$, where g is convex and $g(0) = 0$.

Let $V = (V_0, \dots, V_N)$ be the price process of the Amer. deriv. sec. w/ intrinsic value G .

Then

$$V_0 = \mathbb{E} \left[\frac{G_N^+}{(1+r)^N} \right].$$

same price as a Euro. deriv. w/ payout G_N^+ .

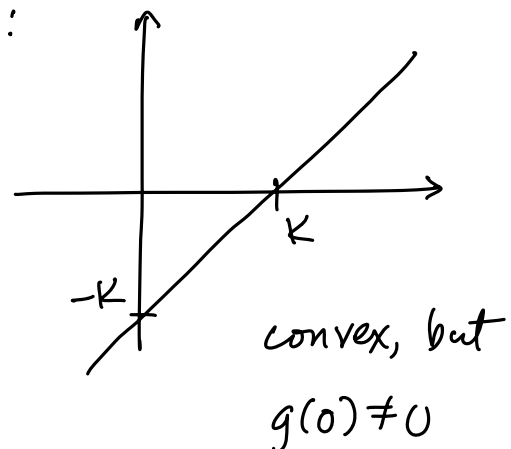
(proof later)

Expl

Amer. call w/ strike price K :

$$G_n = (S_n - K)^+ = g(S_n)$$

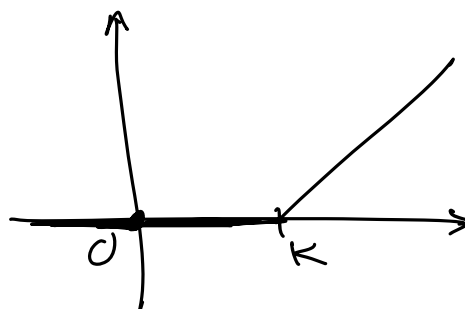
$$g(s) = s - K$$



remodel:

$$G_n = (S_n - K)^+ = g(S_n)$$

$$g(s) = (s - K)^+$$



convex
 $g(0) = 0$

Lemma N4.5.1

Under the assumptions of Thm. 4.5.1,

$\left\{ \frac{G_n^+}{(1+r)^n} \right\}$ is a $\tilde{P}$ -submart.

Pf: adapted ✓

$$g^+(s) := g(s) \vee 0, \quad g^+(0) = 0, \quad g^+(s) \geq 0 \quad \forall s$$

$$g(\lambda a + (1-\lambda)b) \leq \lambda g(a) + (1-\lambda)g(b) \quad \left(\begin{array}{l} \text{because } g \\ \text{is convex} \end{array} \right)$$

$$\leq \lambda g^+(a) + (1-\lambda)g^+(b), \quad \text{and}$$

$$0 \leq \quad "$$

$$\therefore g^+(\lambda a + (1-\lambda)b) \leq \lambda g^+(a) + (1-\lambda)g^+(b) \quad \left(\begin{array}{l} \text{so } g^+ \text{ is} \\ \text{also convex} \end{array} \right)$$

$$a=s, b=0 \Rightarrow g^+(\lambda s) \underset{(*)}{\leq} \lambda g^+(s) \quad \forall s$$

$$\tilde{\mathbb{E}}_n \left[\frac{G_{n+1}^+}{(1+r)^{n+1}} \right] = \frac{1}{(1+r)^n} \tilde{\mathbb{E}}_n \left[\frac{1}{1+r} g^+(S_{n+1}) \right]$$

$$\text{by } (*) \Rightarrow \frac{1}{(1+r)^n} \tilde{\mathbb{E}}_n \left[g^+ \left(\frac{S_{n+1}}{1+r} \right) \right]$$

$$\text{Jensen's ineq.} \Rightarrow \frac{1}{(1+r)^n} g^+ \left(\tilde{\mathbb{E}}_n \left[\frac{S_{n+1}}{1+r} \right] \right)$$

$$= \frac{1}{(1+r)^n} g^+ \left((1+r)^n \tilde{\mathbb{E}}_n \left[\frac{S_{n+1}}{(1+r)^{n+1}} \right] \right)$$

discounted
stock price
is a $\tilde{\mathbb{P}}$ -mart.

$$\stackrel{\text{discounted stock price is a } \tilde{\mathbb{P}}\text{-mart.}}{=} \frac{1}{(1+r)^n} g^+ \left((1+r)^n \cdot \frac{S_n}{(1+r)^n} \right)$$

$$= \frac{1}{(1+r)^n} g^+(S_n) = \frac{G_n^+}{(1+r)^n} \cdot \square$$

Thm N4.5.2 (Thm 4.5.1 is a consequence)

If $\left\{ \frac{G_n^+}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -submart., then

can be
path dependent

$$V_0 = \tilde{\mathbb{E}} \left[\frac{G_N^+}{(1+r)^N} \right].$$

discounted payout
keeps getting better;
no reason to
exercise early.

Pf: $\bar{\tau} := N 1_{\{G_N > 0\}} + \infty 1_{\{G_N \leq 0\}}$

$$\begin{aligned} V_0 &= \max_{\tau \in \mathcal{S}_0} \tilde{\mathbb{E}} \left[1_{\{\tau < \infty\}} \frac{1}{(1+r)^\tau} G_\tau \right] \\ &\geq \tilde{\mathbb{E}} \left[1_{\{\bar{\tau} < \infty\}} \frac{1}{(1+r)^{\bar{\tau}}} G_{\bar{\tau}} \right] \\ &= \tilde{\mathbb{E}} \left[1_{\{G_N > 0\}} \frac{1}{(1+r)^N} G_N \right] = \tilde{\mathbb{E}} \left[\frac{G_N^+}{(1+r)^N} \right]. \end{aligned}$$

Now let $\tau \in \mathcal{S}_0$ be arbitrary. Then

$$\begin{aligned} (*) &= \tilde{\mathbb{E}} \left[1_{\{\tau < \infty\}} \frac{1}{(1+r)^\tau} G_\tau \right] \\ &= \tilde{\mathbb{E}} \left[1_{\{\tau < \infty\}} \frac{1}{(1+r)^{N \wedge \tau}} G_{N \wedge \tau} \right] \\ &\leq \tilde{\mathbb{E}} \left[1_{\{\tau < \infty\}} \frac{1}{(1+r)^{N \wedge \tau}} G_{N \wedge \tau}^+ \right] \\ &\leq \tilde{\mathbb{E}} \left[\frac{1}{(1+r)^{N \wedge \tau}} G_{N \wedge \tau}^+ \right] \\ &\leq \tilde{\mathbb{E}} \left[\frac{1}{(1+r)^N} G_N^+ \right] \end{aligned}$$

optional sampling, since $\left\{ \frac{G_n^+}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -submart

$$\therefore V_0 = \max_{\tau \in \mathcal{T}_0} (V) \leq \tilde{\mathbb{E}} \left[\frac{G_N^+}{(1+r)^N} \right] \cdot \square$$

Ch. 4 summary

trading/consumption strategy: (X_0, Δ, C)

X_0 : initial wealth

$\Delta = (\Delta_0, \dots, \Delta_{N-1})$: portfolio process

$C = (C_0, \dots, C_{N-1})$: consumption process

} adapted

$$\underline{X_{n+1} = (1+r)(X_n - C_n - \Delta_n S_n) + \Delta_n S_{n+1}}$$

$G = (G_0, \dots, G_N)$: intrinsic value process of Amer. deriv. sec.

special case: $G_n = g(S_n)$, non-path-dependent

Stopping time: $\tau: \Omega \rightarrow \{0, \dots, N\} \cup \{\infty\}$

such that $\forall n \in \{0, \dots, N-1\}$, $1_{\{\tau = n\}}$ depends only on $\xi_1, \dots, \xi_n$

Optional sampling theorem

If $\{U_n\}_{n=0}^N$ is a submart. under $\mathbb{P}$ and
(super)

τ is a st. time, then $\{U_{n \wedge \tau}\}_{n=0}^N$ is a

submart. and $E[U_{n \wedge \tau}] \leq E[U_n]$.
(super) ($\geq$)

$$(X_0, \Delta, C) \mapsto X = \{X_0, \dots, X_N\}$$

if $C_n \geq 0 \forall n$, then $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -supermart.
($\leq$) (sub)

Given $W = \{W_0, \dots, W_N\}$ adapted, $\exists (X_0, \Delta, C)$

s.t. $X_n = W_n \forall n$. Namely: (replicating strategy)

$$X_0 = W_0$$

$$\Delta_n = \frac{W_{n+1}(\xi_1, \dots, \xi_n, 1) - W_{n+1}(\xi_1, \dots, \xi_n, -1)}{S_{n+1}(\xi_1, \dots, \xi_n, 1) - S_{n+1}(\xi_1, \dots, \xi_n, -1)}$$

$$C_n = W_n - \frac{1}{1+r} \tilde{E}_n[W_{n+1}]$$

If $\left\{ \frac{W_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -supermart, then $C_n \geq 0 \forall n$.
(sub) ($\leq$)

$$\tilde{E}_n[W_{n+1}] = \tilde{p} W_{n+1}(\xi_1, \dots, \xi_n, 1) + \tilde{q} W_{n+1}(\xi_1, \dots, \xi_n, -1)$$

Given $G = (G_0, \dots, G_N)$, define

$$V_n = \max_{\tau \in \mathcal{T}_n} \tilde{\mathbb{E}}_n \left[\mathbf{1}_{\{\tau < \infty\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

$\uparrow$
 st. times w/ $\tau \geq n$

“price process” of Amer. deriv.

Thm 4.4.3 $\Rightarrow V_N = G_N^+$ and

$$V_n = G_n \vee \left(\frac{1}{1+r} \tilde{\mathbb{E}}_n[V_{n+1}] \right)$$

for $G_n = g(S_n)$: $V_n = v_n(S_n)$, where $v_N(s) = g(s)^+$ and

$$v_n(s) = g(s) \vee \frac{1}{1+r} (\tilde{p} v_{n+1}(us) + \tilde{q} v_{n+1}(ds))$$

Thm 4.4.2 $\Rightarrow$

- $\left\{ \frac{V_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -supermart. (can be hedged)
- $V_n \geq G_n^+$ (is enough to cover the seller)
- if (X_0, Δ, C) is s.t. $C_n \geq 0$ and $X_n \geq G_n^+$,
then $X_n \geq V_n$ (is the most efficient
hedging strategy)

If the price were greater than V_0 , the seller could produce an arbitrage.

hedging strategy when $G_n = g(S_n)$:

$$\delta_n(s) = \frac{V_{n+1}(us) - V_{n+1}(ds)}{us - ds}$$

$$c_n(s) = V_n(s) - \frac{1}{1+r} (\tilde{p} V_{n+1}(us) + \tilde{q} V_{n+1}(ds))$$

$$X_0 = V_0(S_0), \quad \Delta_n = \delta_n(S_n), \quad C_n = c_n(S_n)$$

$$\tau^* = \min \{n : V_n = G_n\}$$

↑ "optimal exercise time"

Thm 4.4.5+ $\Rightarrow$

- τ^* is a maximizer in the defn of V_0
 - $\tau^* < \infty \Rightarrow V_{\tau^*} = G_{\tau^*}$
 $\tau^* = \infty \Rightarrow V_N = 0$
 - $C_n 1_{\xi_n < \tau^*} = 0 \quad \forall n$
-

If price is less than V_0 , buyer can create an arbitrage by using $(-X_0, -\Delta, -C)$ to hedge $-V$, and exercising at time τ^* .

τ^* is optimal because it prevents the seller from consuming (which would be an arbitrage).

$$\left. \begin{array}{l} G_n = g(S_n) \\ g \text{ convex} \\ g(0) = 0 \end{array} \right\} \Rightarrow \left\{ \frac{G_n^+}{(1+r)^n} \right\} \text{ is a } \tilde{P}\text{-submart.}$$

(e.g. Amer. call)

$$\Rightarrow V_0 = \underbrace{\mathbb{E} \left[\frac{G_N^+}{(1+r)^N} \right]}$$

same price as the European version

HW: Ch. 4 : 1, 2, 6(c), 7

Ch. 5 Random Walk

5.1 Introduction

$$\Omega = \Omega_\infty = \{-1, 1\}^\infty$$

$$= \{\omega = (\omega_1, \omega_2, \dots) : \text{each } \omega_n = -1 \text{ or } 1\}$$

$$\xi_n(\omega) = \omega_n$$

Choose $p \in (0, 1)$. Let $q := 1 - p$

$\mathbb{P}$: a prob. meas that makes

$$\xi_1, \xi_2, \dots \text{ iid}$$

with $\mathbb{P}(\xi_n = 1) = p$, $\mathbb{P}(\xi_n = -1) = q$

$$M_0 = 0$$

$$M_n = \sum_{j=1}^n \xi_j$$

$$M = \{M_n\}_{n=0}^\infty$$

↑
stoch. proc., adapted (each M_n depends only on $\xi_1, \dots, \xi_n$)