

that $X_N = W$.

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But in Ch. 2, could we have replicated any wealth process? If $(W_0, \dots, W_N)$ is a stoch. proc., is there a way to trade so that $X_n = W_n \forall n$? No, not in general. When we trade, our discounted wealth is a $\tilde{P}$ -mart. So if $\left\{ \frac{W_n}{(1+r)^n} \right\}$ is not a $\tilde{P}$ -mart., then we can't replicate it.

But can we replicate any such process in the model of Ch. 2? Suppose

$W = (W_0, \dots, W_N)$ is a stoch. proc. s.t.

$\left\{ \frac{W_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -mart. Is there a way

to trade so that $X_n = W_n \forall n$? Yes!

Just replicate W_N . Then, since $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is also a $\tilde{P}$ -mart., we get

$$\begin{aligned}
 X_n &= (1+r)^n \cdot \frac{X_n}{(1+r)^n} = (1+r)^n \tilde{\mathbb{E}}_n \left[\frac{X_N}{(1+r)^N} \right] \\
 &= (1+r)^n \tilde{\mathbb{E}}_n \left[\frac{W_N}{(1+r)^N} \right] = (1+r)^n \cdot \frac{W_n}{(1+r)^n} = W_n.
 \end{aligned}$$

Now, though, in Ch. 4, we have

consumption, so the situation is different.

By Cor. N4.3.2, if we trade w/ nonnegative consumption, then $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -supermart.

But what if $(W_0, \dots, W_N)$ is a stoch. proc.

s.t. $\left\{ \frac{W_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -supermart? Can

we find a way to trade with nonneg.

consumption so that $X_n = W_n \forall n$? Yes!

That's the next theorem. Every process

whose discounted version is a $\tilde{\mathbb{P}}$ -supermart.

can be replicated with nonneg. consumption.

Thm N4.4.2

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Let $W = (W_0, \dots, W_N)$ be an adapted stoch.

proc. Define $X_0 = W_0$,

$$\Delta_n = \frac{W_{n+1}(\xi_1, \dots, \xi_n, 1) - W_{n+1}(\xi_1, \dots, \xi_n, -1)}{S_{n+1}(\xi_1, \dots, \xi_n, 1) - S_{n+1}(\xi_1, \dots, \xi_n, -1)},$$

and

$$C_n = W_n - \frac{1}{1+r} \tilde{E}_n[W_{n+1}].$$

Then the trading strategy (X_0, Δ, C) produces a

wealth process X s.t. $X_n = W_n \forall n$. Moreover,

if $\left\{ \frac{W_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -supermart $(\leq)$, then $C_n \geq 0 \forall n$.
(sub)

Pf: Induction: $X_0 = W_0$ by defn. Suppose

$X_n = W_n$. Then

$$X_{n+1} = (1+r)(X_n - C_n - \Delta_n S_n) + \Delta_n S_{n+1}$$

$$= \Delta_n (\cancel{S_{n+1}} - (1+r)S_n) + (1+r)(\cancel{X_n} - C_n)$$

$\gamma_{n+1} S_n$

$W_n \leftarrow$ inductive step

$$= \Delta_n (\gamma_{n+1} - (1+r)) S_n + (1+r)(W_n - C_n).$$

Now,

$$\begin{aligned}\Delta_n &= \frac{W_{n+1}(\xi_1, \dots, \xi_n, 1) - W_{n+1}(\xi_1, \dots, \xi_n, -1)}{\cancel{S_{n+1}(\xi_1, \dots, \xi_n, 1)} - \cancel{S_{n+1}(\xi_1, \dots, \xi_n, -1)}} \\ &\quad \quad \quad uS_n \quad \quad \quad dS_n \\ &= \frac{W_{n+1}(\xi_1, \dots, \xi_n, 1) - W_{n+1}(\xi_1, \dots, \xi_n, -1)}{(u-d)S_n},\end{aligned}$$

So

$$\begin{aligned}\Delta_n (Y_{n+1} - (1+r)S_n) \\ = (W_{n+1}(\xi_1, \dots, \xi_n, 1) - W_{n+1}(\xi_1, \dots, \xi_n, -1)) \cdot \frac{Y_{n+1} - (1+r)S_n}{u-d}\end{aligned}$$

Also, by the defn of C_n ,

$$\begin{aligned}(1+r)(W_n - C_n) &= \tilde{\mathbb{E}}_n[W_{n+1}] \\ &= \tilde{p} W_{n+1}(\xi_1, \dots, \xi_n, 1) + \tilde{q} W_{n+1}(\xi_1, \dots, \xi_n, -1).\end{aligned}$$

Adding these together gives

$$\begin{aligned}X_{n+1} &= \left(\tilde{p} + \frac{Y_{n+1} - (1+r)S_n}{u-d} \right) W_{n+1}(\xi_1, \dots, \xi_n, 1) \\ &\quad + \left(\tilde{q} - \frac{Y_{n+1} - (1+r)S_n}{u-d} \right) W_{n+1}(\xi_1, \dots, \xi_n, -1).\end{aligned}$$

But $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-(1+r)}{u-d}$, so

$$X_{n+1} = \frac{Y_{n+1}-d}{u-d} W_{n+1}(\xi_1, \dots, \xi_n, 1) \\ + \frac{u-Y_{n+1}}{u-d} W_{n+1}(\xi_1, \dots, \xi_n, -1).$$

When $\xi_{n+1} = 1$, the right-hand side (RHS) equals

$$W_{n+1}(\xi_1, \dots, \xi_n, 1) = W_{n+1}(\xi_1, \dots, \xi_n, \xi_{n+1}) = W_{n+1}.$$

When $\xi_{n+1} = -1$, the RHS equals

$$W_{n+1}(\xi_1, \dots, \xi_n, -1) = W_{n+1}(\xi_1, \dots, \xi_n, \xi_{n+1}) = W_{n+1}.$$

So in any case, the RHS equals W_{n+1} ,

showing that $X_{n+1} = W_{n+1}$. By induction,

$$X_n = W_n \quad \forall n.$$

Now suppose $\left\{ \frac{W_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -supermart. (sub)

Then

$$\frac{1}{1+r} \tilde{\mathbb{E}}_n[W_{n+1}] = (1+r)^n \tilde{\mathbb{E}}_n \left[\frac{W_{n+1}}{(1+r)^{n+1}} \right]$$

$$\leq (1+r)^n \cdot \frac{W_n}{(1+r)^n} = W_n, \\ (\geq)$$

showing that

$$C_n = W_n - \frac{1}{1+r} \tilde{E}_n[W_{n+1}]$$

$$\underset{(\leq)}{\geq} 0. \quad \square$$

Thm 4.4.2 (justifies calling V_n the price,
from seller's POV)

Let $G = (G_0, \dots, G_N)$ be the intrinsic value process of an American derivative security, and $V = (V_0, \dots, V_N)$ its price process. Then

(i) $\left\{ \frac{V_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -supermart, so V can be replicated with nonneg. consumption,

(ii) $V_n \geq G_n^+ \quad \forall n$, and

(iii) if (X_0, Δ, C) is any trading strategy with $C_n \geq 0$, and if the resulting wealth process satisfies $X_n \geq G_n^+ \quad \forall n$, then $X_n \geq V_n \quad \forall n$.

Pf:

(i) adapted ✓

$$\tilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right]$$

$$= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n \left[\max_{\tau \in \mathcal{S}_{n+1}} \tilde{\mathbb{E}}_{n+1} \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-(n+1)}} G_\tau \right] \right]$$

$$= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n \left[\tilde{\mathbb{E}}_{n+1} \left[1_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n-1}} G_{\tau^*} \right] \right]$$

choose $\tau^* \in \mathcal{S}_{n+1}$ that maximizes

$$= \frac{1}{(1+r)^n} \tilde{\mathbb{E}}_n \left[1_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*} \right]$$

$$\leq \frac{1}{(1+r)^n} \max_{\tau \in \mathcal{S}_n} \tilde{\mathbb{E}}_n \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

because $\tau^* \in \mathcal{S}_n$ also

$$= \frac{1}{(1+r)^n} V_n . \checkmark$$

$$(ii) \quad V_n = \max_{\tau \in \mathcal{S}_n} \tilde{\mathbb{E}}_n \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

Let $\tau_1 = n$. Then $\tau_1 \in \mathcal{S}_n$, so

$$V_n \geq \tilde{E}_n \left[1_{\{\tau_1 \leq N\}} \frac{1}{(1+r)^{\tau_1-n}} G_{\tau_1} \right]$$

$$= \tilde{E}_n[G_n] = G_n.$$

Let $\tau_2 = \infty$. Then $\tau_2 \in \mathcal{T}_n$, so

$$V_n \geq \tilde{E}_n \left[1_{\{\tau_2 \leq N\}} \frac{1}{(1+r)^{\tau_2-n}} G_{\tau_2} \right] = 0.$$

Thus $V_n \geq G_n$ and $V_n \geq 0$, which implies

$$V_n \geq \max\{G_n, 0\} = G_n^+. \checkmark$$

(iii) Let (X_0, Δ, C) be a trading strategy with $C_n \geq 0 \forall n$ and suppose that $X_n \geq G_n^+ \forall n$.

By Cor. N4.3.2, $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{P}$ -supermart.

By Thm. 4.3.2, $\left\{ \frac{X_{n \wedge \tau}}{(1+r)^{n \wedge \tau}} \right\}$ is a $\tilde{P}$ -supermart.

Fix n . Let $\tau \in \mathcal{T}_n$ be arbitrary. Then

$$\begin{aligned}
1_{\{\tau \leq N\}} G_\tau &= 1_{\{\tau \leq N\}} G_{N \wedge \tau} \\
&\leq 1_{\{\tau \leq N\}} G_{N \wedge \tau}^+ \\
&\leq G_{N \wedge \tau}^+ \leq X_{N \wedge \tau}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\tilde{E}_n \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right] \\
&= (1+r)^n \tilde{E}_n \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{N \wedge \tau}} G_\tau \right] \\
&\leq (1+r)^n \tilde{E}_n \left[\underbrace{\frac{X_{N \wedge \tau}}{(1+r)^{N \wedge \tau}}}_{\tilde{P}\text{-supermart}} \right] \\
&\leq (1+r)^n \cdot \frac{X_{n \wedge \tau}}{(1+r)^{n \wedge \tau}} \\
&= (1+r)^n \cdot \frac{X_n}{(1+r)^n} = X_n.
\end{aligned}$$


 since $\tau \in \mathcal{S}_n$

Since $\tau \in \mathcal{S}_n$ was arbitrary,

$$\begin{aligned}
V_n &= \max_{\tau \in \mathcal{S}_n} \tilde{E}_n \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right] \\
&\leq X_n. \quad \square
\end{aligned}$$

So $V = (V_0, \dots, V_N)$ is the right price,
 and Thm. N4.4.2 tells us how to replicate it.
 But to apply Thm N4.4.2, we have to be
 able to calculate V_n . The defn of V_n is
 not very good for calculating.

Thm 4.4.3

Let $G = (G_0, \dots, G_N)$ be the intrinsic value process
 of an American derivative security, and $V = (V_0, \dots, V_N)$
 its price process. Then $V_N = G_N^+$ and

$$V_n = G_n \vee \left(\frac{1}{1+r} \tilde{E}_n[V_{n+1}] \right).$$

Pf: Define $Z_N = G_N^+$ and

$$Z_n = G_n \vee \left(\frac{1}{1+r} \tilde{E}_n[Z_{n+1}] \right).$$

We want to show that $V_n = Z_n \quad \forall n$.

We first show that

$$\left\{ \frac{Z_n}{(1+r)^n} \right\} \text{ is a } \mathbb{P}\text{-supermart.} \quad (*)$$

$$Z = (Z_0, \dots, Z_N) \text{ adapted } \checkmark$$

By the defn of Z_n ,

$$Z_n \geq \frac{1}{1+r} \tilde{E}_n[Z_{n+1}].$$

Thus,

$$\tilde{E}_n \left[\frac{Z_{n+1}}{(1+r)^{n+1}} \right] \leq \frac{1}{(1+r)^{n+1}} \cdot (1+r) Z_n = \frac{Z_n}{(1+r)^n},$$

so (*) is true. We can therefore find a trading strategy (X_0, Δ, C) with $C_n \geq 0$ and $X_n = Z_n \forall n$.

We next show that $X_n = Z_n \geq G_n^+ \forall n$.

For $n=N$, we have $Z_N = G_N^+ \checkmark$

Assume true for $n+1$. Then $Z_{n+1} \geq G_{n+1}^+ \geq 0$,

so $\tilde{E}_n[Z_{n+1}] \geq 0$, and hence,

$$Z_n = G_n \vee \left(\frac{1}{1+r} \tilde{E}_n[Z_{n+1}] \right) \geq G_n \vee 0 = G_n^+ \checkmark$$

This proves that $X_n = Z_n \geq G_n^+ \quad \forall n$.

By Thm. 4.4.2, $X_n = Z_n \geq V_n \quad \forall n$.

To finish the proof, we must show that $Z_n \leq V_n \quad \forall n$. By Lemma N4.4.1, $V_N = G_N^+$.

Thus, $Z_N = V_N$. ✓ Assume $Z_{n+1} \leq V_{n+1}$.

Then

$$\begin{aligned} \frac{1}{1+r} \tilde{E}_n[Z_{n+1}] &\leq \frac{1}{1+r} \tilde{E}_n[V_{n+1}] \\ &= (1+r)^n \tilde{E}_n \left[\underbrace{\frac{V_{n+1}}{(1+r)^{n+1}}}_{\tilde{P}\text{-submart. by Thm 4.4.2}} \right] \\ &\leq (1+r)^n \cdot \frac{V_n}{(1+r)^n} = V_n. \end{aligned}$$

Also by Thm. 4.4.2, $V_n \geq G_n^+$. Thus,

$$V_n \geq G_n^+ \vee \left(\frac{1}{1+r} \tilde{E}_n[Z_{n+1}] \right) = Z_n. \quad \square$$

Proof of Thm 4.2.2

Let $G_n = g(S_n)$. Consider the American derivative with intrinsic value process $G = (G_0, \dots, G_N)$. Let

$V = (V_0, \dots, V_N)$ be the price process. We will prove that $V_n = v_n(S_n) \quad \forall n$. For $n=N$, we have $V_N = G_N^+ = g(S_N)^+ = v_N(S_N)$. Suppose $V_{n+1} = v_{n+1}(S_{n+1})$. By Thm. 4.4.3,

$$\begin{aligned}
 V_n &= G_n \vee \left(\frac{1}{1+r} \tilde{E}_n[V_{n+1}] \right) \\
 &= g(S_n) \vee \left(\frac{1}{1+r} \tilde{E}_n[v_{n+1}(S_{n+1})] \right) \\
 &= g(S_n) \vee \left(\frac{1}{1+r} \tilde{E}_n[v_{n+1}(Y_{n+1} S_n)] \right) \\
 &= g(S_n) \vee \left(\frac{1}{1+r} (\tilde{p} v_{n+1}(u S_n) + \tilde{q} v_{n+1}(d S_n)) \right) \\
 &= v_n(S_n). \quad \checkmark
 \end{aligned}$$

Let (X_0, Δ, C) be the strategy from Thm N4.2.2, so that $C_n \geq 0 \quad \forall n$ and $X_n = V_n \quad \forall n$.

$X_0 = V_0 = v_0(S_0)$, it suffices to verify that

$\Delta_n = \delta_n(S_n) \quad \forall n$ and $C_n = c_n(S_n)$ for all n ,

which is straightforward. $\square$

Thm 4.2.2 shows that the time 0 price cannot be greater than V_0 . If the price were $x > V_0$, the seller could sell the derivative for x , pocket $x - V_0$, and use $X_0 = V_0$ to replicate with nonnegative consumption. If the buyer exercises at time τ , the seller will have $V_\tau \geq G_\tau^+$.

The seller will pay the buyer G_τ , keep $V_\tau - G_\tau \geq 0$, and keep any consumption prior to τ .

But what if the price were $x < V_0$?

How can the buyer create an arbitrage from this? Based on what we saw with European derivatives we expect that they would capitalize on the cheap price by using