

## 4.4 General American derivatives

$$\mathcal{S}_n = \{ \tau : \tau \text{ is a st. time and } \tau \geq n \}$$

$$\mathcal{S}_0 = \text{all st. times}$$

$$\mathcal{S}_1 = \text{all nonzero st. times} \quad \left( \begin{array}{l} \text{for any st. time } \tau, \\ \text{either } \tau = 0 \text{ or } \tau \geq 1, \\ \text{i.e. } P(\tau = 0) \in \{0, 1\}. \end{array} \right)$$

$$\mathcal{S}_N = \text{all st. time } \tau \text{ w/ } P(\tau = N \text{ or } \tau = \infty) = 1$$

$$G = (G_0, \dots, G_N) \text{ adapted}$$

$\nwarrow$  intrinsic value process of the Amer. deriv.

Pays  $G_n$  if exercised at time  $n$ .

( $G_n = g(S_n)$  in the previous section)

### Defn N4.4.1

The price process of the Amer. deriv is

$$V = (V_0, \dots, V_N), \text{ where}$$

$$V_n = \max_{\tau \in \mathcal{S}_n} \tilde{E}_n \left[ 1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

$V$  is adapted

$\mathcal{S}_n$  is a finite set. (why?)  
So this is a max

Need theorems to justify calling it a "price".

IDEA:

At time  $n$ , buyer picks  $\tau \in \mathbb{S}_n$  as an exercise strategy.

If  $\tau \leq N$ , buyer will get  $G_\tau$

Present value, at time  $n$ :  $\frac{1}{(1+r)^{\tau-n}} G_\tau$

If  $\tau = \infty$ , buyer will get 0.

So in either case, buyer will get (present value)

$$1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau$$

present value  
of what you'll  
get

In the European case, price is  $\tilde{E}_n \left[ \frac{V_N}{(1+r)^{N-n}} \right]$

So in this case, the "price" is

$$\max_{\tau \in \mathbb{S}_n} \tilde{E}_n \left[ 1_{\{\tau \leq n\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

assumes buyer will choose  $\tau$   
to maximize their value

This doesn't prove it deserves to be called the "price". And the intuition is weak; based on symbolic similarity to European pricing formula.

We will have thus later that justify calling this the "price".

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For each  $n$ , choose  $\tau_n^* \in \mathcal{S}_n$  so that

$$\begin{aligned} V_n &= \max_{\tau \in \mathcal{S}_n} \tilde{E}_n \left[ 1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right] \\ &= \tilde{E}_n \left[ 1_{\{\tau_n^* \leq N\}} \frac{1}{(1+r)^{\tau_n^*-n}} G_{\tau_n^*} \right] \end{aligned}$$

Then  $\tau^* := \tau_0^*$  is called the optimal exercise time.

Need a <sup>↑</sup>thm to justify this name. Also need a formula for  $\tau^*$ .

Lemma N4.4.1

$$V_N = G_N^+$$

Pf:

$$V_N = \max_{\tau \in \mathcal{S}_N} \tilde{E}_N \left[ 1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-N}} G_\tau \right]$$

↖ at time N, we know everything.

$$= \max_{\tau \in \mathcal{S}_N} \left( 1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-N}} G_\tau \right).$$

$$\tau \in \mathcal{S}_N \Rightarrow \{\tau \leq N\} = \{\tau = N\}, \text{ so}$$

$$V_N = \max_{\tau \in \mathcal{S}_N} \left( 1_{\{\tau = N\}} \frac{1}{(1+r)^{\tau-N}} G_\tau \right)$$

$$= \max_{\tau \in \mathcal{S}_N} 1_{\{\tau = N\}} G_N$$

$$G_N \leq G_N^+ \text{ and } G_N^+ \geq 0 \Rightarrow$$

$$1_{\{\tau = N\}} G_N \leq 1_{\{\tau = N\}} G_N^+ \leq G_N^+.$$

$$\text{So } V_N \leq G_N^+.$$

Now let  $\tau = N 1_{\{G_N > 0\}} + \infty 1_{\{G_N \leq 0\}}$ .

Then  $\tau \in \mathcal{S}_N$ , so

$$V_N = \max_{\tau \in \mathcal{S}_N} 1_{\{\tau = N\}} G_N$$

$$\geq 1_{\{\tau = N\}} G_N$$

But  $\tau = N$  iff  $G_N > 0$ , so  $\{\tau = N\} = \{G_N > 0\}$ .

Thus,

$$V_N \geq 1_{\{G_N > 0\}} G_N = G_N^+. \quad \square$$

Expl 4.2.1 (again, different question)

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}$$

Consider an American put w/ strike price  $K=5$ .

Find the price process  $V = (V_0, V_1, V_2)$ .

$$G_n = g(S_n), \quad g(s) = 5 - s$$

$$G_n = 5 - S_n$$

$$V_N = G_N^+$$

$$V_2 = G_2^+ = (5 - S_2)^+ \checkmark$$

$$V_1 = \max_{\tau \in \mathcal{S}_1} \tilde{E}_1 \left[ 1_{\{\tau \leq 2\}} \frac{1}{(1+r)^{\tau-1}} G_\tau \right]$$

↑ st. times that  
are  $\geq 1$

Find the  $\tau$  that maximizes this.

Will learn how to do this later.

time of first tail  
at or after  
time  $n=1$

For now, the answer is  $\tau^* = \min \{n \geq 1 : \xi_n = -1\}$

$$V_1 = \tilde{E}_1 \left[ 1_{\{\tau^* \leq 2\}} \frac{1}{(1+r)^{\tau^*-1}} G_{\tau^*} \right]$$

$$V_1(1) = \tilde{E}_1 \left[ 1_{\{\tau^* \leq 2\}} \frac{1}{(1+r)^{\tau^*-1}} G_{\tau^*} \right](1)$$

$$= \tilde{E} \left[ \underbrace{1_{\{\tau^* \leq 2\}} \frac{1}{(1+r)^{\tau^*-1}} G_{\tau^*}}_U \mid \xi_1 = 1 \right]$$

$$= \tilde{E}[U(1, \xi_2)] = \tilde{p} U(1, 1) + \tilde{q} U(1, -1)$$

$$= \frac{1}{2} U(1, 1) + \frac{1}{2} U(1, -1) = ?$$

$$U(1,1) = ?$$

$$\tau^*(1,1) = \infty \Rightarrow (1,1) \notin \{\tau^* \leq 2\}$$

$$\Rightarrow \mathbb{1}_{\{\tau^* \leq 2\}}(1,1) = 0 \Rightarrow U(1,1) = 0 \checkmark$$

$$U(1,-1) = ?$$

$$\tau^*(1,-1) = 2 \Rightarrow (1,-1) \in \{\tau^* \leq 2\}$$

$$\Rightarrow \mathbb{1}_{\{\tau^* \leq 2\}}(1,-1) = 1$$

$$\Rightarrow U(1,-1) = \frac{1}{(1+r)^{\tau^*(1,-1)-1}} G_{\tau^*(1,-1)}(1,-1)$$

$$= \frac{1}{(1+r)^{2-1}} G_2(1,-1)$$

$$= \frac{4}{5} (5 - \cancel{S_2(1,-1)}^4) = \frac{4}{5} \checkmark$$

$$V_1(1) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$$

$$V_1(-1) = \tilde{\mathbb{E}}[U | \xi_1 = -1] = \tilde{\mathbb{E}}[U(-1, \xi_2)]$$

$$= \tilde{p} U(-1,1) + \tilde{q} U(-1,-1)$$

$$= \frac{1}{2} U(-1,1) + \frac{1}{2} U(-1,-1) = ?$$

$$U(-1, 1) = ?$$

$$\tau^*(-1, 1) = 1 \Rightarrow U(-1, 1) = \frac{1}{(1+r)^{1-1}} G_1(-1, 1)$$

$G_1$  depends only on  $\xi_1$

$$= G_1(-1)$$

$$U(-1, -1) = ?$$

$$\tau^*(-1, -1) = 1 \Rightarrow U(-1, -1) = \frac{1}{(1+r)^{1-1}} G_1(-1, -1)$$

$$= G_1(-1)$$

$$V_1(-1) = G_1(-1) = 5 - \cancel{S_1(-1)}^2 = 3$$

$$V_0 = \max_{\tau \in \mathcal{S}_0} \tilde{\mathbb{E}} \left[ 1_{\{\tau \leq 2\}} \frac{1}{(1+r)^\tau} G_\tau \right]$$

$\uparrow$  all st. times

Find the  $\tau$  that maximizes this.

Will learn how later.

In this example, the same  $\tau^*$  as before maximizes it.

$$V_0 = \tilde{\mathbb{E}} \left[ 1_{\{\tau^* \leq 2\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*} \right]$$

$$= \tilde{\mathbb{E}} \left[ 1_{\{\tau^* \leq 2\}} \left( \frac{4}{5} \right)^{\tau^*} G_{\tau^*} \right] \rightarrow \tilde{u}$$

$$V_0 = \frac{1}{4} \tilde{u}(1,1) + \frac{1}{4} \tilde{u}(1,-1) + \frac{1}{4} \tilde{u}(-1,1) + \frac{1}{4} \tilde{u}(-1,-1)$$

$$= \frac{1}{4} \cdot 0 + \frac{1}{4} \left(\frac{4}{5}\right)^2 G_2(1,-1) + \frac{1}{4} \left(\frac{4}{5}\right) \cancel{G_1(1,1)} + \frac{1}{4} \left(\frac{4}{5}\right) \cancel{G_1(-1,-1)} + \frac{1}{4} \left(\frac{4}{5}\right) G_1(-1)$$

$\gamma^*(1,1) = \infty$        $\gamma^*(-1,1) = 1$        $G_1(-1)$   
 $\gamma^*(1,-1) = 2$        $\gamma^*(-1,-1) = 1$   
 $G_1(-1)$

$$= \frac{4}{25} G_2(1,-1) + \frac{2}{5} G_1(-1)$$

$$= \frac{4}{25} (5 - \cancel{S_2(1,-1)}) + \frac{2}{5} (5 - \cancel{S_2(-1)})$$

4      2

$$= \frac{4}{25} + \frac{6}{5} = \frac{34}{25}$$

$$V_0 = \frac{34}{25} \quad V_1 = \begin{cases} \frac{2}{5} & \text{if } \xi_1 = 1, \\ 3 & \text{if } \xi_1 = -1 \end{cases}$$

$$V_2 = (5 - S_2)^+$$

In the binomial model w/o consumption (Ch.2) we saw we could replicate any terminal wealth. That is, if  $W$  is any random variable, there is a way to trade so

that  $X_N = W$ .

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But in Ch. 2, could we have replicated any wealth process? If  $(W_0, \dots, W_N)$  is a stoch. proc., is there a way to trade so that  $X_n = W_n \forall n$ ? No, not in general. When we trade, our discounted wealth is a  $\tilde{P}$ -mart. So if  $\left\{ \frac{W_n}{(1+r)^n} \right\}$  is not a  $\tilde{P}$ -mart., then we can't replicate it.

But can we replicate any such process in the model of Ch. 2? Suppose

$W = (W_0, \dots, W_N)$  is a stoch. proc. s.t.

$\left\{ \frac{W_n}{(1+r)^n} \right\}$  is a  $\tilde{P}$ -mart. Is there a way

to trade so that  $X_n = W_n \forall n$ ? Yes!

Just replicate  $W_N$ . Then, since  $\left\{ \frac{X_n}{(1+r)^n} \right\}$  is also a  $\tilde{P}$ -mart., we get