

Portfolio at start of Day 1:

$$\text{mm: } \frac{58}{15} \approx \$3.87$$

$$S_1 = 2$$

$$\text{shares: } -\frac{13}{30}$$

Our wealth at start of Day 1:

$$X_1 = \frac{58}{15} + \left(-\frac{13}{30}\right)(2) = \frac{58-13}{15} = \frac{45}{15} = 3 \text{ dollars}$$

Will buyer exercise option or not?

Case 1: Buyer exercises

We owe $g(2) = 5 - 2 = 3$ dollars.

We have exactly three dollars.

Can settle up and go home.

Case 2: Buyer doesn't exercise

$$\Delta_1(-1) = -1$$

$$\text{Buy } -1 - \left(-\frac{13}{30}\right) = \frac{13-30}{30} = -\frac{17}{30} \text{ shares}$$

(i.e. short $\frac{17}{30}$ more shares)

$$\text{getting us } \frac{17}{30}(2) = \frac{17}{15} \text{ dollars}$$

$$\text{In mm: } \frac{58}{15} + \frac{17}{15} = \frac{75}{15} = 5 \text{ dollars}$$

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$$C_1(-1) = 1$$

do whatever you
want this money.
it's "excess"

Withdraw/consume \$1 in the evening of Day 1

Portfolio at end of Day 1:

mm: 4 dollars

$$S_1 = 2$$

shares: -1

- Suppose buyer didn't exercise and stock goes up to $S_2 = 4$

Portfolio on Day 2:

$$\text{mm: } 4\left(\frac{5}{4}\right) = 5 \text{ dollars}$$

$$S_2 = 4$$

shares: -1

Wealth on Day 2:

$$X_2 = 5 + (-1)(4) = 1 \text{ dollar}$$

Just enough to pay buyer who will now exercise.

4.3 Stopping times

Defn 4.3.1 A stopping time is a random variable, τ , taking values in $\{0, 1, 2, \dots, N\} \cup \{\infty\}$ such that for all $n \in \{0, \dots, N-1\}$, the r.v. $\mathbb{1}_{\{\tau=n\}}$ depends only on $\xi_1, \dots, \xi_n$.

- Think of τ like a random alarm that goes off at time τ . If $\tau = \infty$, then the alarm never goes off.

- Can you "hear" the alarm when it goes off?

At time n , you only know $\xi_1, \dots, \xi_n$. Is that enough to know whether $\tau = n$ or not? Yes,

because $\mathbb{1}_{\{\tau=n\}}$ depends only on $\xi_1, \dots, \xi_n$.

- $\mathbb{1}_{\{\tau=0\}}$ depends on nothing, so should be a constant (either 0 or 1; i.e. $P(\tau=0) = 0$ or 1).

Expl

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$N=2$

Imagine I bought a share of stock at time 0. I plan to keep it and sell it at time $N=2$. But ... if, at time 1, it's about to go down, I want to sell it before it does.

My mathematician friend says that's impossible. I say no, I can write down the time I want to sell. I want to sell at time

$$\tau = \begin{cases} 1 & \text{if } \tilde{\epsilon}_2 = -1, \\ 2 & \text{if } \tilde{\epsilon}_2 = 1. \end{cases}$$

She retorts that τ is not a stopping time, so I will not know when τ happens and therefore won't be able to sell at time τ .

Why is τ not a stopping time?

$$1_{\{\tau=0\}} = 0 \checkmark$$

$$1_{\{\tau=1\}}(1,1) = 2, \quad 1_{\{\tau=1\}}(1,-1) = 1,$$

so $1_{\{\tau=1\}}$ depends on ξ_2 .

Not a stopping time.

Expl

$$N=2$$

τ = time of the 1st tail = $\min \{n : \xi_n = -1\}$ ↙ $\min \phi := \infty$

$$\tau(-1,-1) = 1$$

$$\tau(-1,1) = 1$$

$$\tau(1,-1) = 2$$

$$\tau(1,1) = \infty$$

$$\text{GR } \tau = \begin{cases} 1 & \text{if } \xi_1 = -1, \\ 2 & \text{if } \xi_1 = 1, \xi_2 = -1, \\ \infty & \text{if } \xi_1 = 1, \xi_2 = 1. \end{cases}$$

Is τ a stopping time?

$$1_{\{\tau=0\}} = 0 \checkmark$$

$$1_{\{\tau=1\}} = 1_{\{\xi_1 = -1\}} \text{ depends only on } \xi_1, \checkmark$$

Yes, a stopping time.

If $\{U_n\}_{n=0}^N$ is a stoch. proc. and τ is a st. time, then the stopped process is $\{U_{n \wedge \tau}\}_{n=0}^N$.

Expl 4.2.1 continued

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}$$

Consider an American put w/ strike price $K=5$.

Let X be our wealth process when we use the hedging strategy (X_0, Δ, C) from

Thm. 4.2.2. Let

$$\tau = \min\{n: \tilde{X}_n = -1\} = \text{time of 1st tail.}$$

Let $U_n = \frac{X_n}{(1+r)^n}$. Compute both $\{U_n\}$ and $\{U_{n \wedge \tau}\}$.

By Thm. 4.2.2, $X_n = v_n(S_n)$, where the v_n 's are defined recursively in Thm. 4.2.2.

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$$U_n = \frac{1}{(1+r)^n} X_n = \left(\frac{4}{5}\right)^n X_n$$

$$X_0 = v_0(S_0) = v_0(4) \overset{\text{computed earlier (CE)}}{=} 1.36 \Rightarrow U_0 = 1.36$$

$$X_1 = v_1(S_1)$$

$$X_1(1) = v_1(8) \overset{\text{CE}}{=} \frac{8}{5} \Rightarrow U_1(1) = \frac{8}{25} = 0.32$$

$$X_1(-1) = v_1(2) \overset{\text{CE}}{=} 3 \Rightarrow U_1(-1) = \frac{12}{5} = 2.40$$

$$X_2 = v_2(S_2) = (5 - S_2)^+$$

$$X_2(1,1) = v_2(16) = 0 \Rightarrow U_2(1,1) = 0$$

$$X_2(1,-1) = X_2(-1,1) = v_2(4) = 1$$

$$\Rightarrow U_2(1,-1) = U_2(-1,1) = \left(\frac{4}{5}\right)^2 = \frac{16}{25} = 0.64$$

$$X_2(-1,-1) = v_2(1) = 4 \Rightarrow U_2(-1,-1) = \frac{64}{25} = 2.56$$

$$U_0 = 1.36, \quad U_1 = \begin{cases} 0.32 & \text{if } \xi_1 = 1, \\ 2.40 & \text{if } \xi_1 = -1, \end{cases} \quad U_2 = \begin{cases} 0 & \text{if } \alpha = 2, \\ 0.64 & \text{if } \alpha = 1, \\ 2.56 & \text{if } \alpha = 0. \end{cases}$$

$$U_{0\wedge\tau} = U_0 = 1.36$$

$$U_{1\wedge\tau} = U_1 = \text{see above}$$

$\uparrow$
 $\tau \geq 1$

$$U_{2\wedge\tau} = ?$$

$$\begin{aligned} U_{2\wedge\tau}(1,1) &= U_{2\wedge\tau(1,1)}(1,1) = U_{2\wedge\infty}(1,1) \\ &= U_2(1,1) = 0 \end{aligned}$$

$$\begin{aligned} U_{2\wedge\tau}(1,-1) &= U_{2\wedge\tau(1,-1)}(1,-1) = U_{2\wedge 2}(1,-1) \\ &= U_2(1,-1) = 0.64 \end{aligned}$$

$$\begin{aligned} U_{2\wedge\tau}(-1,1) &= U_{2\wedge\tau(-1,1)}(-1,1) = U_{2\wedge 1}(-1,1) \\ &= U_1(-1,1) = 2.40 \end{aligned}$$

$$\begin{aligned} U_{2\wedge\tau}(-1,-1) &= U_{2\wedge\tau(-1,-1)}(-1,-1) = U_{2\wedge 1}(-1,-1) \\ &= U_1(-1,-1) = 2.40 \end{aligned}$$

$$U_{0\wedge\tau} = 1.36, \quad U_{1\wedge\tau} = \begin{cases} 0.32 & \text{if } \xi_1 = 1, \\ 2.40 & \text{if } \xi_1 = -1, \end{cases}$$

$$U_{2\wedge\tau} = \begin{cases} 0 & \text{if } \xi_1 = \xi_2 = 1, \\ 0.64 & \text{if } \xi_1 = 1, \xi_2 = -1, \\ 2.40 & \text{if } \xi_1 = -1. \end{cases}$$

$$U_n = \frac{X_n}{(1+r)^n}$$

$\{U_n\}$ is not a $\tilde{P}$ -mart. because of the consumption.

Recall that $C = (C_0, C_1)$, $C_0 = 0$, $C_1 = \begin{cases} 0 & \text{if } \xi_1 = 1, \\ 1 & \text{if } \xi_1 = -1. \end{cases}$

Under $\tilde{P}$, investor is indifferent between stock and money market. This is what made their discounted wealth a $\tilde{P}$ -mart. before we introduced consumption.

But now, if $\xi_1 = -1$, the change in wealth between $n=1$ and $n=2$ will have a decreasing component that has nothing to do with the stock or the money market.

It turns out $\{U_n\}$ is a $\tilde{P}$ -supermart.

(We'll prove this shortly.)

On the other hand, with τ as in this expt, $\{U_{n\wedge\tau}\}$ is a $\tilde{P}$ -mart.

(We'll prove this later.)

This is because $\{U_{n\wedge\tau}\}$ never "sees" the consumption. If $\xi_i = -1$, then $\tau = 1$, so we never see the change from time $n=1$ to time $n=2$.

Thm N4.3.1

Let (X_0, Δ, C) be a trading strategy. Let X be the wealth process generated by (X_0, Δ, C) .

Define $U = \{U_0, \dots, U_N\}$ by

$$U_n = \frac{X_n}{(1+r)^n} + \sum_{j=0}^{n-1} \frac{C_j}{(1+r)^j}.$$

present value (at 0) of all consumption before time n.

↑ present value (at time 0) of X_n

Then U is a $\tilde{P}$ -mart.

Pf: adapted ✓

$$\tilde{E}_n[U_{n+1}] = \tilde{E}_n \left[\frac{X_{n+1}}{(1+r)^{n+1}} + \sum_{j=0}^n \frac{C_j}{(1+r)^j} \right]$$

← depends only on $\xi_1, \dots, \xi_n$

$$= \frac{1}{(1+r)^{n+1}} \tilde{E}_n[X_{n+1}] + \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= \frac{1}{(1+r)^{n+1}} \tilde{E}_n[(1+r)(X_n - C_n - \Delta_n S_n) + \Delta_n S_{n+1}] + \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= \frac{1}{(1+r)^{n+1}} \left((1+r)(X_n - C_n - \Delta_n S_n) + \Delta_n \overset{(1+r)S_n}{\cancel{\tilde{E}_n[S_{n+1}]}} \right) + \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= \frac{X_n}{(1+r)^n} - \frac{C_n}{(1+r)^n} + \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= \frac{X_n}{(1+r)^n} + \sum_{j=0}^{n-1} \frac{C_j}{(1+r)^j} = U_n. \quad \square$$

Cor N4.3.2

Let (X_0, Δ, C) be a trading strategy. Let

X be the wealth process generated by (X_0, Δ, C) .

If $C_n \geq 0 \quad \forall n$, then $\left\{ \frac{X_n}{(1+r)^n} \right\}$ is a $\tilde{\mathbb{P}}$ -supermart.
($\leq$) (submart.)

Pf:

adapted $\checkmark$

Let U be as in Thm N4.3.1. Then

$$\tilde{\mathbb{E}}_n \left[\frac{X_{n+1}}{(1+r)^{n+1}} \right] = \tilde{\mathbb{E}}_n \left[U_{n+1} - \sum_{j=0}^n \frac{C_j}{(1+r)^j} \right]$$

depends only on $\xi_1, \dots, \xi_n$

$$= \tilde{\mathbb{E}}_n [U_{n+1}] - \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= U_n - \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= \frac{X_n}{(1+r)^n} + \sum_{j=0}^{n-1} \frac{C_j}{(1+r)^j} - \sum_{j=0}^n \frac{C_j}{(1+r)^j}$$

$$= \frac{X_n}{(1+r)^n} - \frac{C_n}{(1+r)^n} \stackrel{\geq 0}{\leq} \frac{X_n}{(1+r)^n} . \quad \square$$

Thm 4.3.2 (Optional sampling - Part I)

Let $\{U_n\}_{n=0}^N$ be a supermart. and τ a

(submart.)

(mart.)

stopping time. Then $\{U_{n \wedge \tau}\}$ is a supermart.
(submart.)
(Pf omitted) (mart.)

Thm 4.3.3 (Optional sampling - Part II)

Let $\{U_n\}_{n=0}^N$ be a supermart. (submart.) (mart.) and τ a

stopping time. Then

$$E[U_{n \wedge \tau}] \underset{(\geq)}{\overset{(\leq)}{=}} E[U_n] \quad \forall n$$

(Pf omitted)

4.4 General American derivatives

$$\mathcal{S}_n = \{ \tau : \tau \text{ is a st. time and } \tau \geq n \}$$

$$\mathcal{S}_0 = \text{all st. times}$$

$$\mathcal{S}_1 = \text{all nonzero st. times} \quad \left(\begin{array}{l} \text{for any st. time } \tau, \\ \text{either } \tau = 0 \text{ or } \tau \geq 1, \\ \text{i.e. } P(\tau = 0) \in \{0, 1\}. \end{array} \right)$$

$$\mathcal{S}_N = \text{all st. time } \tau \text{ w/ } P(\tau = N \text{ or } \tau = \infty) = 1$$

$$G = (G_0, \dots, G_N) \text{ adapted}$$

$\nwarrow$ intrinsic value process of the Amer. deriv.

Pays G_n if exercised at time n .

($G_n = g(S_n)$ in the previous section)

Defn N4.4.1

The price process of the Amer. deriv is

$$V = (V_0, \dots, V_N), \text{ where}$$

$$V_n = \max_{\tau \in \mathcal{S}_n} \tilde{E}_n \left[1_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

V is adapted

$\mathcal{S}_n$ is a finite set. (why?)
So this is a max