

Expl 3.3.2 again (using theorem)

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}, p=\frac{2}{3}$$

$$U(x) = \log x, \quad X_0 = 4$$

Find Δ that maximizes $E[U(X_2)]$.

$$a = 0 > -\infty \checkmark$$

U is strictly incr., strictly concave, cont. diff'ble on $(0, \infty)$ ✓

$$U'(x) = \frac{1}{x} \text{ for } x \in (0, \infty)$$

$$m_0 = U'(0) = \infty, \quad m_\infty = U'(\infty) = 0$$

$$\begin{aligned} I(y) &= (U')^{-1}(y) : [m_\infty, m_0] \rightarrow [a, \infty] \\ &\quad : [0, \infty] \rightarrow [0, \infty] \end{aligned}$$

$$y = U'(x) = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$$I(y) = \frac{1}{y} \text{ for } y \in (0, \infty)$$

$$I(\infty) = 0, \quad I(0) = \infty, \quad I(y) = \infty \quad \forall y < 0$$

$$a_0 = \frac{1}{(1+r)^N} a = 0$$

$$X_0 = 4 > a_0 \checkmark$$

Solve $E[\xi I(\lambda \xi)] = 4$ for $\lambda > 0$ (a soln exists)

$$4 = E[\xi \mathbb{I}(\lambda \xi)] = E[\xi \cdot \frac{1}{\lambda \xi}]$$

$$= E[\frac{1}{\lambda}] = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{4}$$

Define $W^* = \mathbb{I}(\lambda \xi) = \mathbb{I}(\frac{\xi}{4}) = \frac{4}{\xi}$

If we replicate W^* , then we will get

$$X_0 = 4 \quad \text{and} \quad X_2 = W^*, \quad \text{and this will}$$

maximize $E[\log X_2]$.

Calculate W^* so we can replicate:

$$W^* = \frac{4}{\xi} = \frac{4(1+r)^2}{Z}$$

$$W^*(\omega) = 4(1+r)^2 \cdot \frac{P(\omega)}{\tilde{P}(\omega)} = \frac{25}{4} \cdot \frac{P(\omega)}{\tilde{P}(\omega)} = 25 P(\omega)$$

$$W^*(1,1) = 25 \cdot \frac{4}{9} = \frac{100}{9}$$

$$W^*(1,-1) = 25 \cdot \frac{2}{9} = \frac{50}{9}$$

$$W^*(-1,1) = 25 \cdot \frac{2}{9} = \frac{50}{9}$$

$$W^*(-1,-1) = 25 \cdot \frac{1}{9} = \frac{25}{9}$$

3.4 Summary

Radon-Nikodym derivative of $\tilde{P}$ w.r.t. P :

$$Z(\omega) = \frac{\tilde{P}(\omega)}{P(\omega)} = \left(\frac{\tilde{q}}{q}\right)^N \left(\frac{\tilde{p}/\tilde{q}}{p/q}\right)^{\alpha(\omega)} \quad \text{odds of heads}$$

For any r.v. Y , $\tilde{E}[Y] = E[ZY]$

Radon-Nikodym derivative process: $(Z_0, \dots, Z_N)$

$$Z_n = E_n[Z], \quad Z_N = Z, \quad Z_0 = 1$$

$$Z_n(\theta) = \frac{\tilde{P}(A_\theta)}{P(A_\theta)} = \left(\frac{\tilde{q}}{q}\right)^n \left(\frac{\tilde{p}/\tilde{q}}{p/q}\right)^{\alpha(\theta)}$$

$\theta \in \{-1, 1\}^n$ (points to θ)
 $\uparrow$ first n flips match θ (points to A_θ)
 $\uparrow$ # of 1's in θ (points to $\alpha(\theta)$)

If $Y = Y(\xi_1, \dots, \xi_n)$, then

$$\tilde{E}_m[Y] = \frac{1}{Z_m} E_m[Z_n Y] \quad \text{for } m \leq n$$

Taking $m=0$: $\tilde{E}[Y] = E[Z_n Y]$

State price density: $\zeta = \frac{Z}{(1+r)^N}$

If $\omega^* \in \Omega$ and $V_N = 1_{\{\omega^*\}}$,

then $V_0 = \zeta(\omega^*) P(\omega^*)$.

For any V_N , $V_0 = E[\zeta V_N]$

State price density process : $(\zeta_0, \dots, \zeta_N)$

$$\zeta_n = \frac{Z_n}{(1+r)^n}, \quad \zeta_N = \zeta, \quad \zeta_0 = 1$$

For any V_N , $V_n = \frac{1}{\zeta_n} E_n[\zeta V_N]$.

According to our book, a utility function is

a nondecreasing, concave func. $U: (a, \infty) \rightarrow \mathbb{R}$,

where $a \in [-\infty, \infty)$. We say $U(x) = -\infty \quad \forall x \leq a$,

$$U(-\infty) = \lim_{x \rightarrow -\infty} U(x), \text{ and } U(\infty) = \lim_{x \rightarrow \infty} U(x).$$

The book goes on to say they assume U is strictly concave on (a, ∞) .

We assume even more:

- $a \neq -\infty$
- U is strictly increasing on (a, ∞)
- U is continuously diff'ble on (a, ∞)

$$m_\infty = U'(\infty) := \lim_{x \rightarrow \infty} U'(x) \in [0, \infty)$$

$$m_a = U'(a) := \lim_{x \downarrow a} U'(x) \in (m_\infty, \infty]$$

$$I := (U')^{-1}: [m_\infty, m_a] \rightarrow [a, \infty], \quad I(y) := a \quad \forall y > m_a$$

↖ strictly decr., bijections ↗

Optimal investment

Let $X_0 > a_0 := \frac{1}{(1+r)^N} a$ be given.

Solve $E[\ell I(\lambda \ell)] = X_0$ for $\lambda > 0$. (a soln exists)

Define $W^* = I(\lambda \ell)$.

Let (X^*, Δ^*) replicate W^* .

Then $X^* = X_0$ and the strategy (X_0, Δ^*)

produces a final wealth X_N that

maximizes $E[U(X_N)]$.

HW: 3.2, 3.4, 3.6, 3.7

§4.2 Non-path-dependent American derivatives

If $g: (0, \infty) \rightarrow \mathbb{R}$, then a European option to get $g(S_N)$ at time N is modeled by

$$V_N = g(S_N)^+ = g(S_N) \vee 0$$

e.g. call: $g(s) = s - K$, put: $g(s) = K - s$

An American option allows you to get $g(S_n)$ at time n for any n of your choice.

↑ depends only on current stock price for now.
"non-path-dependent"

- Never worth less than the European version.

The difference in value is the early exercise premium

- Sometimes worth the same as the European version; e.g. American call (will see later)

Story:

- American options have become popular on Reddit. People have to exercise by 4pm ET, to give seller time to plan their day.
- New Portfolio 2.0 app:
Cash-flow tab removed. No more integration of derivative securities.

Besides the initial transfer, you can now make one withdrawal each evening after 4pm ET

- can be negative, in which case it's a deposit.
- The amount withdrawn on Day n ($n=0,1,2,\dots,N-1$) is C_n .

$$C = (C_0, C_1, C_2, \dots, C_{N-1}) \leftarrow \text{adapted}$$

$\uparrow$
consumption process

New recursion for wealth process:

X_n : wealth at beginning of Day n

$$X_{n+1} = (1+r)(X_n - C_n - \Delta_n S_n) + \Delta_n S_{n+1}$$

New trading strategy: (X_0, Δ, C)

Thm 4.2.2 (replication of non-path-dependent Amer. options)

Let $g: (0, \infty) \rightarrow \mathbb{R}$ be given. Define $v_n(s) = g(s)^+$ and

$$v_n(s) = g(s) \vee \frac{1}{1+r} (\tilde{p} v_{n+1}(us) + \tilde{q} v_{n+1}(ds)).$$

Let

$$\delta_n(s) = \frac{v_{n+1}(us) - v_{n+1}(ds)}{us - ds}, \text{ and}$$

$$c_n(s) = v_n(s) - \frac{1}{1+r} (\tilde{p} v_{n+1}(us) + \tilde{q} v_{n+1}(ds)).$$

Choose $X_0 = v_0(S_0)$, $\Delta_n = \delta_n(S_n)$, and $C_n = c_n(S_n)$.

Then, for all n ,

$$C_n \geq 0 \quad \text{and} \quad X_n = v_n(S_n).$$

Consequently, $X_n \geq g(S_n)$ for all n .

Pf: Will prove later.

Thm 4.2.2 gives the seller a way to hedge an American option. But is it the cheapest?

Will think about that later.

Expl 4.2.1

$$N=2, S_0=4, u=2, d=\frac{1}{2}, r=\frac{1}{4}$$

Consider an American put w/ strike price $K=5$.

Find the hedging strategy (X_0, Δ, C) from Thm 4.2.2.

$$g(s) = 5 - s$$

$$v_2(s) = (5 - s)^+, \quad \tilde{p} = \tilde{q} = \frac{1}{2}$$

$$v_1(s) = (5 - s) \vee \left(\frac{2}{5} v_2(2s) + \frac{2}{5} v_2(\frac{1}{2}s) \right)$$

$$v_0(s) = (5 - s) \vee \left(\frac{2}{5} v_1(2s) + \frac{2}{5} v_1(\frac{1}{2}s) \right)$$

$$X_0 = v_0(S_0) = v_0(4)$$

$$v_0(4) = 1 \vee \left(\frac{2}{5} v_1(8) + \frac{2}{5} v_1(2) \right)$$

$$v_1(8) = (-3) \vee \left(\frac{2}{5} \cancel{v_2(16)}^0 + \frac{2}{5} \cancel{v_2(4)}^1 \right) = \frac{2}{5}$$

while we're at it:

$$c_1(8) = v_1(8) - \left(\frac{2}{5} \cancel{v_2(16)} + \frac{2}{5} \cancel{v_2(4)} \right) = 0$$

In general, calculate

$$\begin{aligned} v_n(s) &= g(s) \vee \frac{1}{1+r} \left(\cancel{f} v_{n+1}(us) + \cancel{f} v_{n+1}(ds) \right) \\ &= a \vee b \end{aligned}$$

If b is max, $c_n(s) = 0$.

If a is max, $c_n(s) = a - b$.

$$v_1(2) = 3 \vee \left(\frac{2}{5} \cancel{v_2(4)}^1 + \frac{2}{5} \cancel{v_2(1)}^4 \right)$$

$$= 3 \vee \left(\frac{2}{5} + \frac{8}{5} \right) = 3 \vee 2 = 3 \quad \leftarrow \text{different from the Euro version}$$

$$c_1(2) = 3 - 2 = 1$$

$$v_0(4) = 1 \vee \left(\frac{2}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot 3 \right) = 1 \vee \left(\frac{4}{25} + \frac{30}{25} \right)$$

$$= 1 \vee \frac{34}{25} = \frac{34}{25} = 1.36 \quad \left(\text{Euro put worth } 0.96 \text{ at time } 0 \right)$$

$$c_0(4) = 0$$

(130)

$$X_0 = v_0(S_0) = v_0(4) = 1.36$$

$$\begin{aligned}\Delta_0 = \delta_0(4) &= \frac{v_1(8) - v_1(2)}{8 - 2} = \frac{1}{6} \left(\frac{2}{5} - 3 \right) \\ &= \frac{1}{6} \left(\frac{2-15}{5} \right) = -\frac{13}{30}\end{aligned}$$

$$\Delta_1 = \delta_1(S_1)$$

$$\Delta_1(1) = \delta_1(8) = \frac{\cancel{v_2(16)}^0 - \cancel{v_2(4)}^1}{16 - 4} = -\frac{1}{12}$$

$$\Delta_1(-1) = \delta_1(2) = \frac{\cancel{v_2(4)}^1 - \cancel{v_2(1)}^4}{4 - 1} = -1$$

$$C_0 = c_0(4) = 0$$

$$C_1 = c_1(S_1)$$

$$C_1(1) = c_1(8) = 0$$

$$C_1(-1) = c_1(2) = 1$$

$$X_0 = 1.36, \quad \Delta = (\Delta_0, \Delta_1), \quad C = (C_0, C_1)$$

$$\Delta_0 = -\frac{13}{30}$$

$$C_0 = 0$$

$$\Delta_1 = \begin{cases} -\frac{1}{12} & \text{if } \xi_1 = 1, \\ -1 & \text{if } \xi_1 = -1 \end{cases}$$

$$C_1 = \begin{cases} 0 & \text{if } \xi_1 = 1, \\ 1 & \text{if } \xi_1 = -1. \end{cases}$$

Narrative of hedging:

- Sell American put for 1.36
- Make initial deposit $X_0 = 1.36 = \frac{34}{25}$

On Day 0, buy $\Delta_0 = -\frac{13}{30}$ shares

(i.e. short $\frac{13}{30}$ shares)

Withdraw/consume $C_0 = 0$

In mm at end of Day 0:

$$\begin{aligned} \frac{34}{25} - \left(-\frac{13}{30}\right)(4) &= \frac{34}{25} + \frac{26}{15} = \frac{102 + 130}{75} \\ &= \frac{232}{75} \end{aligned}$$

Portfolio at end of Day 0:

$$\text{mm: } \frac{232}{75} \approx \$3.09$$

$$S_0 = 4$$

$$\text{shares: } -\frac{13}{30}$$

- Suppose stock goes down between Days 0 and 1:

$$S_0 = 2, \text{ in mm: } \frac{232}{75} \cdot \frac{5}{4} = \frac{58}{15}$$

Portfolio at start of Day 1:

$$\text{mm: } \frac{58}{15} \approx \$3.87$$

$$S_1 = 2$$

$$\text{shares: } -\frac{13}{30}$$

Our wealth at start of Day 1:

$$X_1 = \frac{58}{15} + \left(-\frac{13}{30}\right)(2) = \frac{58-13}{15} = \frac{45}{15} = 3 \text{ dollars}$$

Will buyer exercise option or not?

Case 1: Buyer exercises

We owe $g(2) = 5 - 2 = 3$ dollars.

We have exactly three dollars.

Can settle up and go home.

Case 2: Buyer doesn't exercise

$$\Delta_1(-1) = -1$$

$$\text{Buy } -1 - \left(-\frac{13}{30}\right) = \frac{13-30}{30} = -\frac{17}{30} \text{ shares}$$

(i.e. short $\frac{17}{30}$ more shares)

$$\text{getting us } \frac{17}{30}(2) = \frac{17}{15} \text{ dollars}$$