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Financial Mathematics

Review of probability (§§ 2.1-2.3)

Notation: $a \vee b = \max \{a, b\}$, $a, b \in \mathbb{R}$
 $a \wedge b = \min \{a, b\}$
 $a^+ = a \vee 0$
 $a^- = -(a \wedge 0)$

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases} \quad (A \text{ is a set})$$

↑
indicator function of A

finite probability space : $(\Omega, \mathcal{P})$

Ω is a finite set, $\mathcal{P}: 2^\Omega \rightarrow [0, 1]$

↑ power set of Ω , i.e.
set of all subsets of Ω

$\mathcal{P}$ must satisfy the axioms of probability

$\mathcal{P}$ is called a probability measure

Ω is called the sample space

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 $\omega \in \Omega$
 ↖ outcome
 $A \subset \Omega$
 ↖ event

 shorthand: $P(\omega) = P(\{\omega\})$

$$P(A) = \sum_{\omega \in A} P(\omega)$$

need only specify these values

random variable: $X: \Omega \rightarrow [-\infty, \infty]$
 usually just $\mathbb{R}$ here

if $X(\omega) = c \quad \forall \omega \in \Omega$, then X is a constant r.v. (can think of constants as r.v.s)

$R(X)$ = range of X

$X = Y$ means $X(\omega) = Y(\omega) \quad \forall \omega \in \Omega$

mass function of X : $p_X(x) = P(X=x) \quad \forall x \in \mathbb{R}$
 ↖ also called the distribution of X

Expected value: $E^P[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$
 ↖ sometimes we change P

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Useful property: $E^P[1_A] = P(A)$

Variance: $\text{Var}_P(X) = E^P[(X - E^P[X])^2]$

Alternate formulas:

$$E^P[X] = \sum_{x \in R(X)} x P(X=x)$$

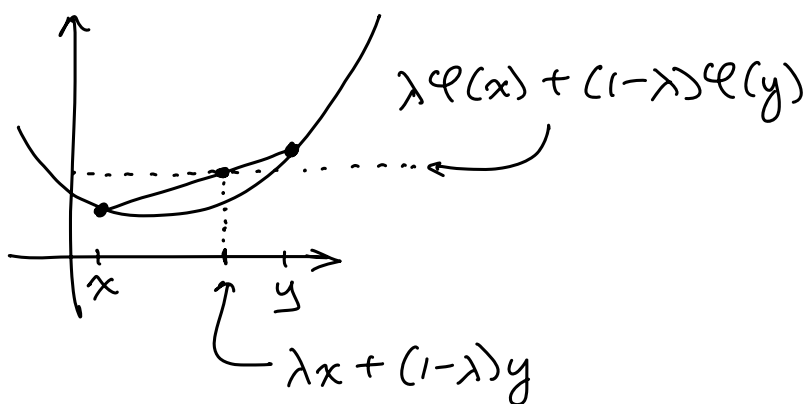
$$\text{Var}_P(X) = E^P[X^2] - (E^P[X])^2$$

$\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is (strictly) convex if $\forall x < y, \forall \lambda \in (0,1)$,

$$\varphi(\lambda x + (1-\lambda)y) \underset{(<)}{\leq} \lambda \varphi(x) + (1-\lambda)\varphi(y)$$

(means φ'' exists and is continuous)

If $\varphi \in C^2(\mathbb{R})$, then φ is convex iff $\varphi'' \geq 0$.



Jensen's inequality:

$$\varphi \text{ convex} \Rightarrow \varphi(E^P[X]) \leq E^P[\varphi(X)].$$

Expl: Use $\varphi(x) = x^2$. Then $(E^P[X])^2 \leq E^P[X^2]$.

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Conditioning on an event :

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$E^P[X|A] = \sum_{x \in R(X)} x P(X=x|A)$$

} needs
 $P(A) > 0$

Conditioning on a r.v. :

same as above, since
 $\{Y=y\}$ is an event

$$E^P[X|Y=y] = \sum_{x \in R(X)} x P(X=x|Y=y) \text{ if } P(Y=y) > 0$$

$$E^P[X|Y] = g(Y), \text{ where}$$

$$g(y) = E^P[X|Y=y]$$

$$P(A|Y) := E^P[1_A|Y]$$

Y can even be a vector.

e.g. $Y = (Y_1, Y_2, Y_3)$

$$E^P[X|Y] = E^P[X|Y_1, Y_2, Y_3] = ?$$

First compute

$$\begin{aligned} g(y) &= g(y_1, y_2, y_3) \\ &= E^P[X | Y_1 = y_1, Y_2 = y_2, Y_3 = y_3], \end{aligned}$$

then plug in:

$$E^P[X | Y_1, Y_2, Y_3] = g(Y_1, Y_2, Y_3).$$

Coin toss space

Flip a coin N times. Heads = 1
Tails = -1

sample space $\rightarrow \Omega = \{-1, 1\}^N$

$\omega \in \Omega$, $\omega = (\omega_1, \dots, \omega_N)$, each $\omega_j \in \{-1, 1\}$

$$\alpha(\omega) := |\{n : \omega_n = 1\}|$$

$\uparrow$ r.v. (# of heads)

$N - \alpha(\omega)$: # of tails

$$\sum_n \omega_n = \sum_n \omega_n$$

$\uparrow$ r.v. (result of n^{th} flip)

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Pick a $p \in [0, 1]$
 $\hookrightarrow$ probability of heads on one flip

$$P^p(\omega) := p^{\alpha(\omega)} (1-p)^{N-\alpha(\omega)}$$

$\hookrightarrow$ a probability measure

(Ω_N, P^p) is a finite probability space that models flipping a coin N times with probability p of heads on each flip, and the flips are independent.

Expl

$$\Omega = \Omega_3, \quad X = \alpha, \quad Y = 3 - \alpha$$

$\hookrightarrow$ # of heads $\quad \hookrightarrow$ # of tails

$$P = P^{2/3}$$

distributions of X and Y on (Ω, P) :

t	0	1	2	3
$P_X(t)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$
$P_Y(t)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

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$$\tilde{P} = P^{1/2}$$

distributions of X and Y on $(\Omega, \tilde{P})$:

t	0	1	2	3
$P_X(t)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$P_Y(t)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E^P[X] = \sum_{k=0}^3 k P(X=k) = \frac{6}{27} + 2 \cdot \frac{12}{27} + 3 \cdot \frac{8}{27} = 2$$

$$E^{\tilde{P}}[X] = \sum_{k=0}^3 k \tilde{P}(X=k) = \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$$

$X \stackrel{d}{=} Y$ under $\tilde{P}$, $X \not\stackrel{d}{=} Y$ under P
 $\uparrow$ equal in distribution

Subtle notation in coin toss space

$$\xi_n(\omega) = \omega_n, \quad Y \text{ any r.v.}$$

$$Y(\omega) = Y(\omega_1, \dots, \omega_N)$$

$$Y(\omega) = Y(\xi_1(\omega), \dots, \xi_N(\omega))$$

r.v. $\nearrow Y = Y(\xi_1, \dots, \xi_N)$
 $\nwarrow$ function of coin flips
 Y is both!

e.g. $Y = E^P[X|\xi_1]$

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$$Y = g(\xi_1) \quad \text{where} \quad g(x) = E^P[X|\xi_1 = x]$$

$$g(1) = E^P[X|\xi_1 = 1]$$

$$g(-1) = E^P[X|\xi_1 = -1]$$

But Y and g are actually the same!

So we can write

$$Y(1) = E^P[X|\xi_1 = 1]$$

$$Y(-1) = E^P[X|\xi_1 = -1]$$

or even

$$\begin{aligned} E^P[X|\xi_1](1) &= E^P[X|\xi_1 = 1] \\ E^P[X|\xi_1](-1) &= E^P[X|\xi_1 = -1] \end{aligned}$$

← This is the notation we will use. If the above confuses you, just focus on understanding this notation.

Another example:

$$E^P[X|\xi_1, \xi_2](1, -1) = E^P[X|\xi_1 = 1, \xi_2 = -1]$$

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Yet another example :

$$E^P[X | \xi_1, \xi_2](1, \xi_2) = g(1, \xi_2),$$

$$\text{where } g(x_1, x_2) = E^P[X | \xi_1 = x_1, \xi_2 = x_2]$$

A model for stock prices

Unrealistic. Studying it to learn basic mathematical concepts

IDEA

Time is discrete.

Possible times: $0, 1, 2, \dots, N$

S_n : price of stock at time n

p : probability stock price goes up between times n and $n+1$

$$q := 1 - p$$

u : up factor ($S_{n+1} = uS_n$ when stock goes up)

d : down factor ($S_{n+1} = dS_n$ when stock goes down)

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HOW TO MODEL THIS?(i) Choose $N \in \mathbb{N}$.(ii) Choose $S_0 > 0$.(iii) Choose $p \in (0, 1)$.(iv) Choose u and d so that $0 < d < u$.Let $\Omega = \Omega_N$ and $\mathbb{P} = \mathbb{P}^p$.For $1 \leq n \leq N$, define

$$S_n := \begin{cases} u S_{n-1} & \text{if } \xi_n = 1, \\ d S_{n-1} & \text{if } \xi_n = -1. \end{cases}$$

Sometimes it helps to separate out the random factor.

$$Y_n := \begin{cases} u & \text{if } \xi_n = 1, \\ d & \text{if } \xi_n = -1 \end{cases} \quad \text{and} \quad S_n = Y_n S_{n-1}.$$

This is called the binomial model.NOTE: $S_n = S_n(\xi_1, \dots, \xi_n) \leftarrow S_n$ depends only on the first n coin flips.

Properties of conditional expectation on coin toss space

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$N \in \mathbb{N}$, $p \in (0,1)$, $\Omega = \Omega_N$, $P = P^p$,

X a r.v

notation: $E_0^P[X] = E^P[X]$

$$E_n^P[X] = E^P[X | \xi_1, \dots, \xi_n]$$

linearity:

$$E_n^P[aX + bY] = aE_n^P[X] + bE_n^P[Y]$$

factor out what's known:

If $X = X(\xi_1, \dots, \xi_n)$, then

$$E_n^P[XY] = X E_n^P[Y]$$

special case of last rule:

$$E_N^P[X] = X$$

rule for independence:

if $X = X(\xi_{n+1}, \dots, \xi_N)$, then

$$E_n^P[X] = E^P[X]$$

rule for nesting:

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if $n \leq m$, then

$$E_n^P[E_m^P[X]] = E_m^P[E_n^P[X]] = E_n^P[X]$$

Jensen's inequality:

if φ is convex, then

$$\varphi(E_n^P[X]) \leq E_n^P[\varphi(X)]$$

Computing conditional expectations

$$E_n^P[X] = g(\xi_1, \dots, \xi_n), \text{ where}$$
$$g(x_1, \dots, x_n) = E^P[X(x_1, \dots, x_n, \xi_{n+1}, \dots, \xi_N)]$$

Why? Break it down:

$$E_n^P[X] = E^P[X | \xi_1, \dots, \xi_n] = g(\xi_1, \dots, \xi_n),$$

where

$$g(x_1, \dots, x_n) = E^P[X | \xi_1 = x_1, \dots, \xi_n = x_n]$$

Remember that $X = X(\xi_1, \dots, \xi_N)$, so (13)

$$g(x_1, \dots, x_n)$$

$$= E^P[X(\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_N) \mid \xi_1 = x_1, \dots, \xi_n = x_n]$$

can be random vectors

Theorem from 4.1.3: If $\overset{\downarrow}{U}$ and $\overset{\downarrow}{V}$ are independent,
then

$$E^P[h(U, V) \mid U = u] = E^P[h(u, V)]$$

Using this theorem, we get

$$g(x_1, \dots, x_n) = E^P[X(x_1, \dots, x_n, \xi_{n+1}, \dots, \xi_N)]$$

and that gives use the formula.

Expl

in binomial model : $E_n^P[S_{n+1}] = ?$

$$E_n^P[S_{n+1}] = E_n^P[Y_{n+1} S_n] \quad (\text{defn of } S_{n+1})$$

$$= S_n E_n^P[Y_{n+1}] \quad (\text{factor out what's known})$$

$$= S_n E^P[Y_{n+1}] \quad (\text{rule for independence})$$

$$= S_n(u P(\overset{p}{\cancel{Y_{n+1}=u}}) + d P(\overset{q}{\cancel{Y_{n+1}=d}}))$$

$$= S_n(pu + qd)$$

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$$\boxed{E_n^P[S_{n+1}] = (pu + qd)S_n}$$

In the next 3 examples, we use the binomial model with

$$N=3, S_0=4, u=2, d=\frac{1}{2}, p=\frac{2}{3}$$

Expl 1 $E_1^P[S_2] = ?$

Soln 1 : Use the last example:

$$\begin{aligned} E_1^P[S_2] &= (pu + qd)S_1 \\ &= \left(\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot \frac{1}{2}\right) S_1 \\ &= \left(\frac{4}{3} + \frac{1}{6}\right) S_1 = \frac{9}{6} S_1 = \boxed{1.5 S_1} \end{aligned}$$

Soln 2 Use the definition:

$$E_1^P[S_2] = E^P[S_2 | \xi_1] = g(\xi_1), \text{ where}$$

$$g(x) = E^P[S_2 | \xi_1 = x]$$

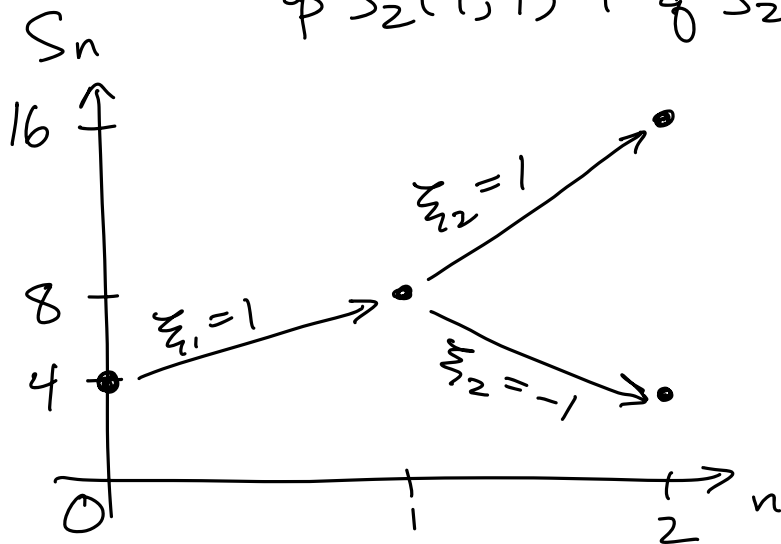
$g(x) = ?$ Need to find $g(1)$ and $g(-1)$. (15)

$$g(1) = E^P[S_2 | \xi_1 = 1]$$

$$= E^P[S_2(\xi_1, \xi_2) | \xi_1 = 1]$$

Then from 4/13 $\Rightarrow E^P[S_2(1, \xi_2)]$

$$= p S_2(1, 1) + q S_2(1, -1)$$



$$g(1) = \frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 4 = \frac{32 + 4}{3} = 12$$

$$g(-1) = E^P[S_2(\xi_1, \xi_2) | \xi_1 = -1]$$

$$= E^P[S_2(-1, \xi_2)]$$

$$= p S_2(-1, 1) + q S_2(-1, -1)$$

$$= \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 1 = \frac{9}{3} = 3$$

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Another way to write this:

$$E_1^P[S_2](1) = E^P[S_2 | \xi_1 = 1] = 12$$

$$E_1^P[S_2](-1) = E^P[S_2 | \xi_1 = -1] = 3$$

Putting it together:

$$E_1^P[S_2] = \begin{cases} 12 & \text{if } \xi_1 = 1, \\ 3 & \text{if } \xi_1 = -1. \end{cases}$$

But...

$$S_1 = \begin{cases} 8 & \text{if } \xi_1 = 1, \\ 2 & \text{if } \xi_1 = -1. \end{cases}$$

$$\text{So } E_1^P[S_2] = \boxed{1.5 S_1}$$

Expl 2 $E_1^P[S_1 S_2](1) = ?$

Soln 1

$$E_1^P[S_1 S_2] = S_1 E_1^P[S_2] \quad (\text{factor out what's known})$$

$$E_1^P[S_1 S_2](1) = \cancel{S_1(1)} \cancel{E_1^P[S_2](1)}$$

8 12 (from last expl)

$$= \boxed{96}$$

Soln 2

$$\begin{aligned}
E_1^P[S_1, S_2](1) &= E^P[S_1, S_2 \mid \xi_1 = 1] \\
&= E^P[S_1(\xi_1) S_2(\xi_1, \xi_2) \mid \xi_1 = 1] \\
&= E^P[\overset{8}{S_1}(1) S_2(1, \xi_2)] \\
&= 8 E^P[S_2(1, \xi_2)] \\
&= 8(p S_2(1, 1) + q S_2(1, -1)) \\
&= 8\left(\frac{2}{3} \cdot 16 + \frac{1}{3} \cdot 4\right) = 8 \cdot 12 = \boxed{96}
\end{aligned}$$

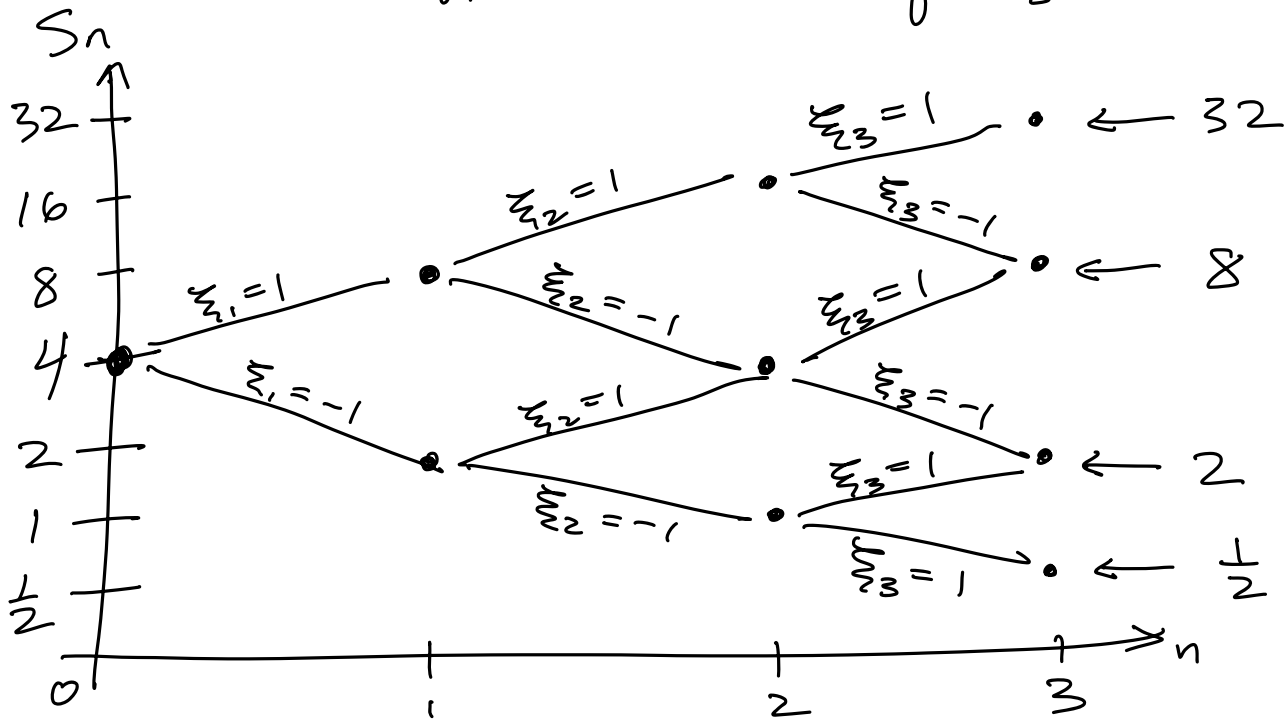
Expl 3 $E_1^P[E_2^P[S_3]](1) = ?$ Soln 1

$$\begin{aligned}
E_1^P[E_2^P[S_3]] &= E_1^P[S_3] \quad (\text{rule for nesting}) \\
E_1^P[E_2^P[S_3]](1) &= E_1^P[S_3](1) \\
&= E^P[S_3 \mid \xi_1 = 1] \\
&= E^P[S_3(\xi_1, \xi_2, \xi_3) \mid \xi_1 = 1] \\
&= E^P[S_3(1, \xi_2, \xi_3)]
\end{aligned}$$

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$$= p^2 S_3(1, 1, 1) + pq S_3(1, 1, -1)$$

$$+ qp S_3(1, -1, 1) + q^2 S_3(1, -1, -1)$$



$$E_1^P[E_2^P[S_3]](1)$$

$$= \frac{4}{9} \cdot 32 + \frac{2}{9} \cdot 8 + \frac{2}{9} \cdot 8 + \frac{1}{9} \cdot 2$$

$$= \frac{128 + 32 + 2}{9} = \frac{162}{9} = \boxed{18}$$

Solu 2

$$X := E_2^P[S_3], \quad X = X(\xi_1, \xi_2)$$

$$E_1^P[E_2^P[S_3]](1) = E_1^P[X](1)$$

$$= E^P[X | \xi_1 = 1]$$

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$$= E^P[X(\xi_1, \xi_2) | \xi_1 = 1]$$

$$= E^P[X(1, \xi_2)]$$

$$= p X(1, 1) + q X(1, -1)$$

$$= \frac{2}{3} X(1, 1) + \frac{1}{3} X(1, -1)$$

$$X(1, 1) = E_2^P[S_3](1, 1)$$

$$= E^P[S_3 | \xi_1 = 1, \xi_2 = 1]$$

$$= E^P[S_3(\xi_1, \xi_2, \xi_3) | \xi_1 = 1, \xi_2 = 1]$$

$$= E^P[S_3(1, 1, \xi_3)]$$

$$= p S_3(1, 1, 1) + q S_3(1, 1, -1)$$

$$= \frac{2}{3} \cdot 32 + \frac{1}{3} \cdot 8 = \frac{72}{3} = 24$$

$$X(1, -1) = E_2^P[S_3](1, -1)$$

$$= E_2^P[S_3(\xi_1, \xi_2, \xi_3) | \xi_1 = 1, \xi_2 = -1]$$

$$= E_2^P[S_3(1, -1, \xi_3)]$$

$$= p S_3(1, -1, 1) + q S_3(1, -1, -1)$$

$$= \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2 = \frac{18}{3} = 6$$

Put it together:

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$$E_1^P[E_2^P[S_3]](1) = \frac{2}{3} \cdot 24 + \frac{1}{3} \cdot 6$$
$$= 16 + 2 = \boxed{18}$$

Ch. 1 The binomial model

"Portfolio" app

MM BROKER

WELCOME

☐ - day plan

OK

"money market"

MM BROKER

BALANCE: \$ 0

ONE-TIME TRANSFER

AMOUNT: \$

BANK INFO:

OK

can be negative

negative allowed, 0 also allowed

required

MM BROKER

TRADE ABC, INC.

SHARES OWNED: 0

PRICE/SHARE: \$ 4

QUANTITY:

BUY

can be negative, need not be integers

- MM earns interest (charges same interest when negative)
- Share purchases use money from MM