

# The logic of random variables

## Expl

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

## SOL'N #1

### Events:

$\Omega$  = "Doors 1, 2, 3 are the only doors,  
and  
Doors 1, 2, 3 are distinct,  
and  
Monty won't show you the car,  
and  
Monty won't open Door 1."

$A_k$  = "The car is behind Door  $k$ ." ( $k=1, 2, 3$ )

$B$  = "Monty opens Door 3."

### Givens:

1.  $P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

2.  $P(B|A_1) = \frac{1}{2}$

### Goal:

$P(A_1|B) = ?$

Sample space:

Truth table, assuming  $\Omega$  is true:

|               | $A_1$ | $A_2$ | $A_3$ | $B$ |
|---------------|-------|-------|-------|-----|
| $\omega_{13}$ | T     | F     | F     | T   |
| $\omega_{12}$ | T     | F     | F     | F   |
| $\omega_{23}$ | F     | T     | F     | T   |
| $\omega_{32}$ | F     | F     | T     | F   |

$$\Omega = \{\omega_{13}, \omega_{12}, \omega_{23}, \omega_{32}\}$$

Solution:

$$A_1 = \{\omega_{13}, \omega_{12}\}, \quad A_2 = \{\omega_{23}\}, \quad A_3 = \{\omega_{32}\}$$

$$B = \{\omega_{13}, \omega_{23}\}$$

$$p_{ij} = P(\{\omega_{ij}\})$$

Given 1  $\Rightarrow$

$$p_{23} = p_{32} = \frac{1}{3}, \quad p_{13} + p_{12} = \frac{1}{3}$$

Given 2  $\Rightarrow$

$$P(B|A_1) = \frac{P(B \cap A_1)}{P(A_1)} = \frac{P(\{\omega_{13}\})}{P(\{\omega_{13}, \omega_{12}\})}$$

$$= \frac{p_{13}}{p_{13} + p_{12}} = \frac{1}{2}$$

$$\Rightarrow 2p_{13} = p_{13} + p_{12} \Rightarrow p_{13} = p_{12}$$

$$\therefore p_{13} = \frac{1}{6} \text{ and } p_{12} = \frac{1}{6}$$

$$\text{So } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{\omega_{13}\})}{P(\{\omega_{13}, \omega_{23}\})}$$

$$= \frac{p_{13}}{p_{13} + p_{23}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} \cdot \frac{6}{6} = \frac{1}{1+2} = \boxed{\frac{1}{3}}$$

## SOL'N #2

Symbols:

← (these are actually called "constant symbols")

$d_1, d_2, d_3$  : the doors

$C$  : the door with the car behind it

$M$  : the door that Monty shows you

Background info:

$$\Omega = " \forall x (x = d_1 \text{ or } x = d_2 \text{ or } x = d_3)$$

and

$$d_1 \neq d_2 \text{ and } d_1 \neq d_3 \text{ and } d_2 \neq d_3$$

and

$$M \neq C$$

and

$$M \neq d_1$$

Givens:

$$1. P(C=d_1) = P(C=d_2) = P(C=d_3) = \frac{1}{3}$$

$$2. P(M=d_3 | C=d_1) = \frac{1}{2}$$

Goal:

$$P(C=d_1 | M=d_3) = ?$$

Sample space:

Outcomes are no longer rows in truth tables.  
They are functions assigning values to symbols (a.k.a. "structures")

$$\omega_{13}: \begin{array}{c|ccccc} s & d_1 & d_2 & d_3 & C & M \\ \hline \omega(s) & 1 & 2 & 3 & 1 & 3 \end{array}$$
$$\omega_{12}: \begin{array}{c|ccccc} s & d_1 & d_2 & d_3 & C & M \\ \hline \omega(s) & 1 & 2 & 3 & 1 & 2 \end{array}$$
$$\omega_{23}: \begin{array}{c|ccccc} s & d_1 & d_2 & d_3 & C & M \\ \hline \omega(s) & 1 & 2 & 3 & 2 & 3 \end{array}$$
$$\omega_{32}: \begin{array}{c|ccccc} s & d_1 & d_2 & d_3 & C & M \\ \hline \omega(s) & 1 & 2 & 3 & 3 & 2 \end{array}$$

these are the  
only possibilities,  
assuming  $\Omega$   
is true.

$$\Omega = \{\omega_{13}, \omega_{12}, \omega_{23}, \omega_{32}\}$$

Can also think of symbols as functions mapping  
outcomes to values. E.g.  $C(\omega_{23}) = \omega_{23}(C) = 2$

Solution:

$$\begin{aligned} "C = d_1" &= \text{set of outcomes in which it is true} \\ &= \{\omega : C(\omega) = d_1(\omega)\} \\ &= \{\omega : C(\omega) = 1\} \quad \left( \begin{array}{l} \text{because } d(\omega) = 1 \\ \text{in every outcome} \end{array} \right) \\ &= \{\omega_{13}, \omega_{12}\} \end{aligned}$$

$$\begin{aligned} "M = d_3" &= \{\omega : M(\omega) = d_3(\omega)\} \\ &= \{\omega : M(\omega) = 3\} \\ &= \{\omega_{13}, \omega_{23}\} \end{aligned}$$

Can do this for any sentence involving our symbols.

$$\text{E.g. , } "C = d_2" = \{\omega : C(\omega) = 2\} = \{\omega_{23}\}$$

$$"C = d_3" = \{\omega : C(\omega) = 3\} = \{\omega_{32}\}$$

$$p_{ij} = P(\{\omega_{ij}\})$$

Given 1  $\Rightarrow$

$$P(C = d_1) = P(\{\omega_{13}, \omega_{12}\}) = p_{13} + p_{12} = \frac{1}{3}$$

$$P(C = d_2) = P(\{\omega_{23}\}) = p_{23} = \frac{1}{3}$$

$$P(C = d_3) = P(\{\omega_{32}\}) = p_{32} = \frac{1}{3}$$

Given 2  $\Rightarrow$

$$P(M = d_3 | C = d_1) = P(\{\omega_{13}, \omega_{23}\} | \{\omega_{13}, \omega_{12}\})$$

$$= \frac{P(\{\omega_{13}\})}{P(\{\omega_{13}, \omega_{12}\})} = \frac{p_{13}}{p_{13} + p_{12}} = \frac{1}{2}$$

As before, this gives  $p_{13} = p_{12} = \frac{1}{6}$ , and so

$$P(C=d_1 | M=d_3)$$

$$= P(\{\omega_{13}, \omega_{12}\} | \{\omega_{13}, \omega_{23}\})$$

$$= \frac{P(\{\omega_{13}\})}{P(\{\omega_{13}, \omega_{23}\})} = \frac{p_{13}}{p_{13} + p_{23}} = \dots = \boxed{\frac{1}{3}}$$


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The symbols we will use to make our events are:

1. Symbols for real numbers
2. All the usual symbols and notation for working with real numbers (e.g.,  $<$ ,  $+$ , etc.)
3. Symbols that we introduce into specific problems, usually capital letters near the end of the alphabet, such as  $X, Y, Z$ .

Only the symbols in Group 3 will have different values in different outcomes, and we call them "random variables".

## Notes

1. Whether outcomes are rows in a truth table or functions assigning values to symbols, they are still the logically possible "states of the world" given our assumptions  $\Omega$ . The "randomness" is our uncertainty about which state is the real one, and  $P$  is our model of that uncertainty.
2. It would be better to call random variables "uncertain constants", but the terminology is fixed by historical momentum.