7.4 Covariance, Variance of Sums, and Correlation

already covered

- X, Y indep $\Rightarrow E[XY] = E[X]E[Y]$
- $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
- $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$
- $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$
- $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$
- $\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + \cancel{\sum\sum_{i \neq j} \mathrm{Cov}(X_i, X_j)}$

$$2\sum\sum_{i<j} \mathrm{Cov}(X_i, X_j)$$

Special Case:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)$$

If $X_1, \ldots, X_n$ are uncorrelated, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

Example 4a Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2, and as in Example 2c, let $\overline{X} = \sum_{i=1}^{n} X_i/n$ be the sample mean. The quantities $X_i - \overline{X}, i = 1, \ldots, n$, are called *deviations*, as they equal the differences between the individual data and the sample mean. The random variable

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

is called the *sample variance*. Find (a) $\text{Var}(\overline{X})$ and (b) $E[S^2]$.

(a) $\text{Var}(\overline{X}) = \text{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)$

$= \frac{1}{n^2}\text{Var}\left(\sum_{j=1}^{n} X_j\right)$

$= \frac{1}{n^2}\sum_{j=1}^{n}\text{Var}(X_j)$ (b/c the X_j's are indep., and ∴ uncorrelated)

$= \frac{1}{n^2}\sum_{j=1}^{n}\sigma^2$

$= \frac{1}{n^2}\cdot n\sigma^2$

$= \boxed{\frac{\sigma^2}{n}}$

(b) $$E[S^2] = E\left[\frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2\right]$$

$$= \frac{1}{n-1}\sum_{j=1}^{n} E\left[(X_j - \overline{X})^2\right]$$

$$E\left[(X_j - \overline{X})^2\right] = E\left[X_j^2 - 2X_j\overline{X} + \overline{X}^2\right]$$

$$= E[X_j^2] - 2E[X_j\overline{X}] + E[\overline{X}^2]$$

$$E[X_j^2] = Var(X_j) + (E[X_j])^2$$

$$= \sigma^2 + \mu^2$$

$$E[\overline{X}^2] = Var(\overline{X}) + (E[\overline{X}])^2$$

($E[\overline{X}] = \mu$ from a previous expl)

$$= \frac{\sigma^2}{n} + \mu^2$$

$$E[X_j\overline{X}] = E\left[X_j \cdot \frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E[X_iX_j]$$

$$i \neq j \Rightarrow E[X_iX_j] = E[X_i]E[X_j] = \mu^2$$

$$i = j \Rightarrow E[X_iX_j] = E[X_j^2] = \sigma^2 + \mu^2$$

$$E[X_i X_j] = \frac{1}{n}\left(\sigma^2 + \mu^2 + (n-1)\mu^2\right)$$

$$= \frac{\sigma^2 + n\mu^2}{n} = \frac{\sigma^2}{n} + \mu^2$$

$$E[(X_i - \bar{X})^2] = \sigma^2 + \mu^2 - 2\left(\frac{\sigma^2}{n} + \mu^2\right) + \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$

$$E[S^2] = \frac{1}{\cancel{n-1}} \sum_{j=1}^{n} \frac{\cancel{n-1}}{n}\sigma^2$$

$$= \frac{1}{n} \cdot n\sigma^2 = \boxed{\sigma^2}$$

7.5 Conditional Expectation

7.5.1 Definitions

$$E[X \mid Y=y] = \begin{cases} \sum_{x \in R(X)} x\, P(X=x \mid Y=y) & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

$E[X \mid Y=y]$ is a function of y

Example 5b Suppose that the joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y} \qquad 0 < x < \infty,\ 0 < y < \infty$$

Compute $E[X|Y = y]$.

X cont:

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$$

From Expl 6.5a:

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} e^{-x/y} & \text{if } y>0 \text{ and } x>0, \\ 0 & \text{if } y>0 \text{ and } x\le 0, \end{cases}$$

If $y \le 0$, $E[X|Y=y]$ is undef.

For $y>0$,

$$E[X|Y=y] = \int_0^{\infty} x \cdot \frac{1}{y} e^{-x/y}\, dx$$

integration by parts

$$= -x e^{-x/y} \Big|_{x=0}^{x=\infty} + \int_0^{\infty} e^{-x/y}\, dx$$

$$= \cancel{0 - 0} - y e^{-x/y} \Big|_{x=0}^{x=\infty}$$

$= 0 - (-y) = y$

$$\boxed{E[X|Y=y] = y \quad \text{if} \quad y > 0}$$

According to $f_{X|Y}(x|y)$, if $y>0$, then $X|Y=y \sim \text{Exp}\left(\frac{1}{y}\right)$, so it's no surprise that $E[X|Y=y]=y$

$$E[X|Y] = \begin{cases} \text{First compute } E[X|Y=y]. \\ \text{This gives you a function of } y. \\ \text{Plug } Y \text{ into that function} \end{cases}$$

$E[X|Y]$ is a random variable

Expl X, Y jointly cont., $X \sim \text{Gamma}(4,1)$,

$$f_{Y|X}(y|x) = \begin{cases} \frac{3(x-y)^2}{8x^3} & \text{if } x>0,\ |y|<x, \\ 0 & \text{if } x>0,\ |y| \geq x. \end{cases}$$

Find (a) $E[Y|X=x]$, and (b) $E[Y|X]$.

(a) Y cont.

$$E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$$

$E[Y|X=x]$ is undef. if $x \le 0$

limits are b/c $|y|<x$ means $-x<y<x$

For $x>0$:

$$E[Y|X=x] = \int_{-x}^{x} y \cdot \frac{3(x-y)^2}{8x^3}\, dy$$

$$= \frac{3}{8x^3} \int_{-x}^{x} y(x^2-2xy+y^2)\, dy$$

$$= \frac{3}{8x^3} \int_{-x}^{x} (\underbrace{x^2y}_{\text{odd func. of } y} - \underbrace{2xy^2}_{\text{even}} + \underbrace{y^3}_{\text{odd}})\, dy$$

symmetric interval of integration

$$= \frac{3}{8x^3} \cdot 2 \int_{0}^{x} (-2xy^2)\, dy$$

$$= -\frac{\cancel{3}}{\cancel{8}_2 x^3} \cdot \cancel{4} \left(\frac{1}{\cancel{3}} x y^3 \Big|_{y=0}^{y=x} \right)$$

$$= -\frac{1}{2x^3} \cdot x^4 = -\frac{x}{2}$$

$$\boxed{E[Y \mid X = x] = -\frac{x}{2} \quad \text{if} \quad x > 0}$$

undefined if $x \leq 0$

(b) Plug X into the function $-\frac{x}{2}$:

$$\boxed{E[Y \mid X] = -\frac{X}{2}}$$

Since $X > 0$, we don't have to worry about the undefined part. This will always happen.

$E[X \mid Y] = g(Y)$, where $g(y) = E[X \mid Y = y]$

Warning: $g(Y) \neq E[X \mid Y = Y]$

Have to compute $E[X \mid Y = y]$ to get an actual function.

Definitions: $P(A \mid X=x) = E[1_A \mid X=x]$

$$P(A \mid X) = E[1_A \mid X]$$

Law of total probability:

$$\boxed{E[X] = E[E[X \mid Y]]}$$

$E[E[X \mid Y]] = E[g(Y)]$, where

$$g(y) = E[X \mid Y=y]$$

will be a sum if Y is discrete,
integral if Y is continuous

Y discrete:

$$E[X] = \sum_{y \in R(Y)} E[X \mid Y=y]\, P(Y=y)$$

Y continuous:

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y=y]\, f_Y(y)\, dy$$

Versions for probabilities:

(original law of total probability)

$$P(A) = \sum_{y \in R(Y)} P(A \mid Y=y)\, P(Y=y) \quad \text{if } Y \text{ discrete}$$

$$P(A) = \int_{-\infty}^{\infty} P(A \mid Y=y)\, f_Y(y)\, dy$$

(If X, Y indep, drop the conditioning and turn Y into y.)

$$\boxed{\begin{array}{l}\text{If } X, Y \text{ \underline{indep.}, then} \\ \quad E[f(X,Y) \mid Y=y] = E[f(X,y)]\end{array}}$$

Special cases:

- X, Y indep $\Rightarrow E[X \mid Y] = E[X]$
- X, Y indep $\Rightarrow$

$$P((X,Y) \in D \mid Y=y) = P((X,y) \in D)$$

<u>Other helpful facts</u>

- $E[cX \mid Y] = c\,E[X \mid Y]$
- $E[X+Y \mid Z] = E[X \mid Z] + E[Y \mid Z]$

- Can pull out known quantities:

$$E[X f(Y) \mid Y] = f(Y)\, E[X \mid Y]$$

Special case: $E[f(Y) \mid Y] = f(Y)$

- With nested conditions, the smaller information dominates:

$$\left.\begin{aligned} E[E[X \mid f(Y)] \mid Y] &= E[X \mid f(Y)] \\ E[E[X \mid Y] \mid f(Y)] &= E[X \mid f(Y)] \end{aligned}\right\}$$ observing $f(Y)$ tells us less than observing Y

Special case:

$$\left.\begin{aligned} E[E[Z \mid X, Y] \mid X] &= E[Z \mid X] \\ E[E[Z \mid X] \mid X, Y] &= E[Z \mid X] \end{aligned}\right\}$$ X is a function of (X, Y)

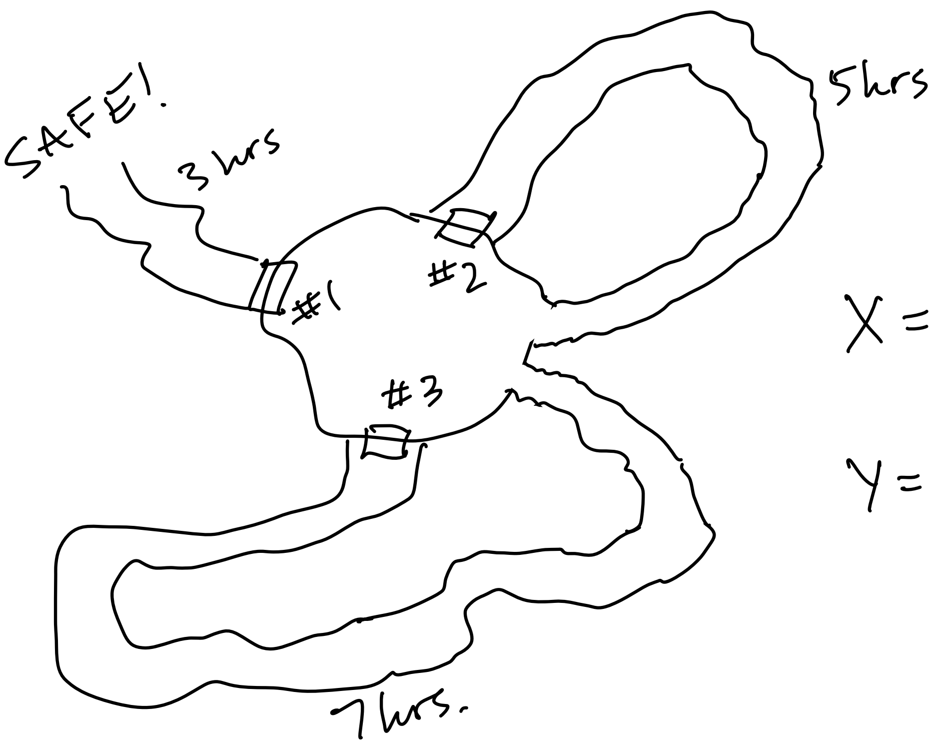
- $X \geq 0 \Rightarrow E[X \mid Y] \geq 0$

- $E[f(X,Y) \mid Y = y] = E[f(X, y) \mid Y = y]$

(doesn't require independence)

7.5.2 Computing expectations by conditioning

Example 5c A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?



X = # of hours he spends in the mine.

Y = # of door he initially chooses

$E[X] = ?$

$$E[X] = E[E[X|Y]]$$

Y is discrete

$$\overset{\downarrow}{=} \sum_{y \in R(Y)} E[X|Y=y]\, p_Y(y)$$

$$R(Y) = \{1, 2, 3\}$$

$$p_Y(1) = P(Y=1) = \frac{1}{3}$$

$$p_Y(2) = P(Y=2) = \frac{1}{3}$$

$$p_Y(3) = P(Y=3) = \frac{1}{3}$$

$$E[X] = \frac{1}{3} E[X|Y=1] + \frac{1}{3} E[X|Y=2] + \frac{1}{3} E[X|Y=3]$$

$E[X \mid Y=1] = 3$ (if he initially picks door #1, he reaches safety in 3 hrs.)

$E[X \mid Y=2] = 5 + E[X]$

(if he initially picks door #2, he wastes 5 hrs and is right back in the same situation)

$E[X \mid Y=3] = 7 + E[X]$

So...

$$E[X] = \frac{1}{3} \cdot 3 + \frac{1}{3}(5 + E[X]) + \frac{1}{3}(7 + E[X])$$

$$E[X] = 1 + \frac{5}{3} + \frac{7}{3} + \frac{2}{3}E[X]$$

$$E[X] = 5 + \frac{2}{3}E[X]$$

$$\frac{1}{3}E[X] = 5 \Rightarrow E[X] = \boxed{15}$$

<u>Expl</u> Generate $U \sim \text{Unif}(0,1)$ and obtain a number p. Build a coin w/ prob. of heads p. Flip it 3 times. Compute the expected # of heads.

N = # of heads in 1st 3 flips

$$E[N] = E[E[N|U]]$$

U is cont., so

$$E[E[N|U]] = \int_{-\infty}^{\infty} E[N|U=p]\, f_U(p)\, dp$$

$$f_U(p) = \begin{cases} 1 & \text{if } 0 < p < 1, \\ 0 & \text{o.w.} \end{cases}$$

$$N|U=p \sim \text{Binom}(3,p)$$

$$E[N|U=p] = 3p$$

$$E[N] = \int_0^1 3p \cdot 1\, dp = \frac{3}{2} p^2 \Big|_{p=0}^{p=1} = \boxed{\frac{3}{2}}$$

Next expl needs this thm: If $X_n \geq 0$ for all n, then

$$E\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} E[X_n]$$

Example 5d **Expectation of a sum of a random number of random variables**

Suppose that the number of people entering a department store on a given day is a random variable with mean 50. Suppose further that the amounts of money spent by these customers are independent random variables having a common mean of \$8. Finally, suppose also that the amount of money spent by a customer is also independent of the total number of customers who enter the store. What is the expected amount of money spent in the store on a given day?

N = # of people that enter

X_j = \$ spent by j^{th} person

$E[N] = 50, \quad E[X_j] = 8$

$N, X_1, X_2, X_3, \ldots$ indep.

T = total \$ spent in the store

$E[T] = ?$

$$T = \sum_{j=1}^{N} X_j$$

$$E[T] = E\left[\sum_{j=1}^{N} X_j\right]$$

Can we do this? $\longrightarrow$ $\cancel{E\left[\sum_{j=1}^{N} X_j\right] = \sum_{j=1}^{N} E[X_j]}$

NO!! (why not?)

$$\underbrace{\cancel{E\left[\sum_{j=1}^{N} X_j\right]}}_{\text{like all expected values, this is a number}} = \underbrace{\cancel{\sum_{j=1}^{N} E[X_j]}}_{\text{but this is a random variable! In fact, it equals } \delta N.}$$

We can't do it because there's a random variable in the limits of summation.

Soln 1

$$E[T] = E\left[\sum_{j=1}^{N} X_j\right] = E\left[E\left[\sum_{j=1}^{N} X_j \,\middle|\, N\right]\right]$$

$$E\left[\sum_{j=1}^{N} X_j \,\middle|\, N\right] = g(N), \text{ where}$$

$$g(n) = E\left[\sum_{j=1}^{N} X_j \,\middle|\, N=n\right]$$

N and $X_1, X_2, X_3, \ldots$ are independent. So drop the conditioning and turn all the N's to the left of the vertical bar into n's

$$g(n) = E\left[\sum_{j=1}^{N} X_j \,\middle|\, N=n\right]$$

$$= E\left[\sum_{j=1}^{n} X_j\right]$$

now there's no random variable in the limits of summation, so it's okay to do this.

$$= \sum_{j=1}^{n} \underbrace{\cancel{E[X_j]}}_{8}$$

$$= 8n$$

$$E\left[\sum_{j=1}^{N} X_j \,\middle|\, N\right] = g(N) = 8N$$

and

$$E[T] = E\left[\sum_{j=1}^{N} X_j\right] = E\left[E\left[\sum_{j=1}^{N} X_j \,\middle|\, N\right]\right]$$

$$= E[8N] = 8\, \cancel{E[N]}^{50} = \boxed{400}$$

Soln 2

$$T = \sum_{j=1}^{N} X_j = \sum_{j=1}^{\infty} \underbrace{X_j 1_{\{j \le N\}}}$$

$= X_j$ for $1 \le j \le N$

$= 0$ for $j > N$

This is another way of getting rid of the random variable in the limits of summation.

$$E[T] = E\left[\sum_{j=1}^{\infty} X_j 1_{\{j \le N\}}\right]$$

normally, you can't do this. We're using a theorem here that's not in our book

$$= \sum_{j=1}^{\infty} E[X_j 1_{\{j \le N\}}]$$

X_j and N are indep.

$$= \sum_{j=1}^{\infty} \underbrace{\cancel{E[X_j]}}_{8} E[1_{\{j \le N\}}]$$

$$= 8 \sum_{j=1}^{\infty} P(N \ge j)$$

remember this formula? It works because $R(N) = \{0, 1, 2, \dots\}$

$$= 8 \underbrace{\cancel{E[N]}}_{50}$$

$$= \boxed{400}$$

Example 5j Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0, 1) random variables. Find $E[N]$ when

$$N = \min\left\{n: \sum_{i=1}^{n} U_i > 1\right\}$$

A trick: for $0 \le x \le 1$,

$$N(x) = \min\left\{n : \sum_{j=1}^{n} U_j > x\right\}$$

We want $E[N(1)]$.

$$E[N(x)] = E\big[E[N(x) \mid U_1]\big]$$

U_1 is cont.

$$\overset{\downarrow}{=} \int_{-\infty}^{\infty} E[N(x) \mid U_1 = y]\, f_{U_1}(y)\, dy$$

$$= \int_0^1 E[N(x) \mid U_1 = y] \cdot 1 \, dy$$

if $y > x$ and $U_1 = y$, then $N(x) = 1$, so

$$E[N(x) \mid U_1 = y] = 1 \text{ when } y > x$$

if $y < x$ and $U_1 = y$, then

$$N(x) = 1 + \overbrace{\min\left\{n : \sum_{j=1}^{n} U_{j+1} > x - y\right\}}^{\text{has the same distribution as } N(x-y)},$$

so $E[N(x) \mid U_1 = y] = 1 + E[N(x-y)]$ when $y < x$.

Putting it together,

$$E[N(x) \mid U_1 = y] = \begin{cases} 1 & \text{if } y > x, \\ 1 + E[N(x-y)] & \text{if } y < x. \end{cases}$$

So

$$E[N(x)] = \int_0^1 E[N(x) \mid U_1 = y]\, dy$$

$$= \int_0^x (1 + E[N(x-y)])\, dy + \int_x^1 1\, dy$$

$$= x + \int_0^x E[N(x-y)]\, dy + (1-x)$$

$$E[N(x)] = 1 + \int_0^x E[N(x-y)]\, dy$$

At first glance, this doesn't look helpful. But more tricks!

$$m(x) = E[N(x)]$$

$$m(x) = 1 + \int_0^x m(x-y)\, dy \quad \left\{ \begin{array}{l} u = x - y \\ du = -dy \\ y = 0 \to u = x \\ y = x \to u = 0 \end{array} \right.$$

$$m(x) = 1 - \int_x^0 m(u)\,du$$

$$m(x) = 1 + \int_0^x m(u)\,du$$

How to solve this?

STEP 1: Plug in $x=0$:

$$\text{(i)} \quad m(0) = 1 + 0 = 1$$

STEP 2: Differentiate this

$$\text{(ii)} \quad m'(x) = m(x)$$

(i) and (ii) are called an initial value problem. You learn how to solve these in a diff. eqns. class. A complete solution, without any shorthand symbol pushing, is here:

$$m'(x) = m(x)$$

$$\frac{m'(x)}{m(x)} = 1$$

$$\int_0^x \frac{m'(t)}{m(t)}\,dt = \int_0^x 1\,dt$$

$$\boxed{\begin{array}{l} u = m(t) \\ du = m'(t)\,dt \end{array}}$$

$$\int_{\cancel{m(0)}\ 1}^{m(x)} \frac{1}{u}\,du = x$$

$$\log u \Big|_{u=1}^{u=m(x)} = x$$

$$\log(m(x)) - \overset{0}{\cancel{\log 1}} = x$$

$$m(x) = e^x$$

So ...

$$E[N] = E[N(1)] = m(1) = \boxed{e}$$

Example 4.7 *Let X have density*

$$f_X(x) = \begin{cases} cx^{-1} & \text{if } 0.5 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y \sim \mathrm{Exp}(1)$. *Assume* X *and* Y *are independent. Find* $E[X^Y]$.

$c = ?$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{\frac{1}{2}}^{1} \frac{c}{x}\,dx$$

$$= c \log x \Big|_{x=\frac{1}{2}}^{x=1} = c\left(0 - \log\tfrac{1}{2}\right)$$

$$= c \log 2$$

$$c = \frac{1}{\log 2} \checkmark$$

<u>Soln 1</u> (condition on X)

$$E[X^Y] = E[E[X^Y \mid X]]$$

X is cont.

$$= \int_{-\infty}^{\infty} E[X^Y \mid X = x]\, f_X(x)\,dx$$

$$= \int_{\frac{1}{2}}^{1} \underline{E[X^Y \mid X = x]} \cdot \frac{1}{\log 2} \cdot \frac{1}{x}\,dx$$

remember for later that $\frac{1}{2} < x < 1$ here

$$E[\underline{X}^Y \mid \underline{X = x}] = E[x^Y]$$

X and Y are indep. So,

- drop conditioning
- turn X into x

$$E[x^Y] = \int_{-\infty}^{\infty} x^y f_Y(y)\, dy$$

$$= \int_0^{\infty} x^y e^{-y}\, dy$$

okay b/c $\frac{1}{2} < x < 1$

$$= \int_0^{\infty} (e^{\log x})^y e^{-y}\, dy$$

$$= \int_0^{\infty} e^{(\log x - 1)y}\, dy$$

Is this improper integral convergent?

$$\log x - 1 \overset{?}{<} 0 \iff \log x \overset{?}{<} 1$$

$$\iff x \overset{?}{<} e \text{, yes.}$$

$$E[x^Y] = \frac{1}{\log x - 1} e^{(\log x - 1)y} \Big|_{y=0}^{y=\infty}$$

$$= \frac{1}{\log x - 1}(0 - 1)$$

$$= \frac{1}{1 - \log x}$$

So

$$E[X^Y] = \int_{\frac{1}{2}}^{1} \frac{1}{1-\log x} \cdot \frac{1}{\log 2} \cdot \frac{1}{x}\, dx$$

$u = \log x$

$du = \frac{1}{x} dx$

$$= \frac{1}{\log 2} \int_{\cancel{\log \frac{1}{2}}\ -\log 2}^{\cancel{\log 1}\ 0} \frac{1}{1-u}\, du$$

$$= \frac{1}{\log 2} \left(-\log(1-u) \Big|_{u=-\log 2}^{u=0} \right)$$

$$= \frac{1}{\log 2} \left(0 - \left(-\log(1+\log 2) \right) \right)$$

$$= \boxed{\frac{\log(1+\log 2)}{\log 2}}$$

Soln 2 (condition on Y)

$$E[X^Y] = E[E[X^Y \mid Y]]$$

Y is cont.

$$= \int_{-\infty}^{\infty} E[X^Y \mid Y=y] f_Y(y)\, dy$$

$$= \int_0^{\infty} E[X^Y \mid Y=y]\, e^{-y}\, dy$$

$$= \int_0^{\infty} E[X^y]\, e^{-y}\, dy$$

remember that $y > 0$ here

$$E[X^y] = \int_{-\infty}^{\infty} x^y f_X(x)\, dx$$

$$= \int_{\frac{1}{2}}^{1} x^y \cdot \frac{1}{\log 2} \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{\log 2} \int_{\frac{1}{2}}^{1} x^{y-1}\, dx$$

okay b/c $y > 0$.

$$= \frac{1}{\log 2} \left(\frac{1}{y} x^{y} \Big|_{x=\frac{1}{2}}^{x=1} \right)$$

$$= \frac{1}{\log 2} \left(\frac{1}{y} - \frac{1}{y} \cdot \left(\frac{1}{2}\right)^{y} \right)$$

$$= \frac{1}{\log 2} \cdot \frac{1 - 2^{-y}}{y}$$

$$E[X^Y] = \int_0^\infty \frac{1}{\log 2} \cdot \frac{1 - 2^{-y}}{y} \cdot e^{-y}\, dy$$

$$= \frac{1}{\log 2} \int_0^\infty \frac{1 - 2^{-y}}{y} e^{-y}\, dy$$

How to integrate this ??

(either give up and use soln 1, or be stubborn and keep going...)

$$2^{-y} = e^{-(\log 2) y} = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n (\log 2)^n y^n$$

$$= 1 - (\log 2) y + \frac{(\log 2)^2}{2!} y^2 - \cdots$$

$$1 - 2^{-y} = (\log 2)\, y - \frac{(\log 2)^2}{2!} y^2 + \frac{(\log 2)^3}{3!} y^3 - \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{n-1} (\log 2)^n y^n$$

$$\frac{1 - 2^{-y}}{y} = \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{n-1} (\log 2)^n y^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} y^n$$

$$E[X^Y] = \frac{1}{\log 2} \int_0^{\infty} e^{-y} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} y^n \right) dy$$

$$= \frac{1}{\log 2} \int_0^{\infty} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} y^n e^{-y} \right) dy$$

$$= \frac{1}{\log 2} \sum_{n=0}^{\infty} \left(\int_0^{\infty} (-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} y^n e^{-y}\, dy \right)$$

$$= \frac{1}{\log 2} \sum_{n=0}^{\infty} \left((-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} \int_0^{\infty} y^n e^{-y}\, dy \right)$$

have to remember defn and properties of gamma func.

$$= \frac{1}{\log 2} \sum_{n=0}^{\infty} \left((-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} \Gamma(n+1) \right)$$

$$= \frac{1}{\log 2} \sum_{n=0}^{\infty} \left((-1)^n \frac{(\log 2)^{n+1}}{(n+1)!} n! \right)$$

$$= \frac{1}{\log 2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (\log 2)^{n+1} = ?$$

Try to build a series like this out of geom. series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

$$\Big\downarrow \text{ integrate}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\log(1-x) + C, \quad |x| < 1$$

Try $x = 0$ to find C:

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = -\log(1-x) + C$$

$$0 + 0 + 0 + \cdots = -\cancel{\log(1)} + C$$

$$0$$

$$0 = C$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\log(1-x), \quad |x| < 1$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} (-x)^{n+1} = -\log(1-(-x)), \quad |x| < 1$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{n+1} = -\log(1+x), \quad |x| < 1$$

$$(-1)\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{n+1} = \log(1+x), \quad |x| < 1$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{n+1} x^{n+1} = \log(1+x), \quad |x| < 1$$

$$(-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n(1) = (-1)^n$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n+1} x^{n+1} = \log(1+x), \quad |x| < 1$$

This will work.

$$E[X^Y] = \frac{1}{\log 2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (\log 2)^{n+1}$$

$$= \frac{1}{\log 2} \log(1 + \log 2)$$

$$= \boxed{\frac{\log(1+\log 2)}{\log 2}}$$

7.5.3 Computing probabilities by conditioning

Example 4.4 *Joe's TV Shop sells expensive electronic equipment. Some customers come to Joe's, browse for a while, and then leave. Some customers (Joe hopes) actually buy something before they leave. It is known that older customers are more likely to buy something than younger customers. In fact, it is known that when a customer who is x years old enters Joe's TV shop, they will buy something with probability $x/(20 + x)$. Let X denote the age of the next customer to enter Joe's TV Shop. Assume that $X \sim \mathrm{Unif}(20, 40)$. What is the probability that this customer buys something?*

A = "The customer buys something."

$P(A) = ?$, X = age of the customer (years)

$$P(A \mid X = x) = \frac{x}{20+x}$$

$$P(A) = E[P(A \mid X)]$$

b/c X is cont.

$$\stackrel{}{=} \int_{-\infty}^{\infty} P(A \mid X=x)\, f_X(x)\, dx$$

$$f_X(x) = \begin{cases} \frac{1}{20} & 20 < x < 40, \\ 0 & \text{o.w.} \end{cases}$$

$$P(A) = \int_{20}^{40} \frac{x}{20+x} \cdot \frac{1}{20}\, dx$$

Technically, this step is partial fractions, but it's easy to do in your head: $\frac{x}{20+x} = \frac{x+20-20}{x+20} = 1 - \frac{20}{x+20}$

$$= \frac{1}{20} \int_{20}^{40} \left(1 - \frac{20}{20+x}\right) dx$$

$$= \frac{1}{20} \left(x - 20 \log(x+20) \right) \Big|_{x=20}^{x=40}$$

$$= \frac{1}{20} \left(40 - 20 \log 60 - 20 + 20 \log 40 \right)$$

$$= \frac{1}{20} \left(20 + 20 \log\left(\tfrac{40}{60}\right) \right)$$

$$= 1 + \log\left(\tfrac{2}{3}\right)$$

$$= \boxed{1 - \log\left(\tfrac{3}{2}\right)}$$

when possible, I like the argument of log to be >1, so it's clearer to my eye that I'm subtracting a positive number.

Example 4.6 *Let X and Y be independent with $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\mu)$. Find $P(X < Y)$.*

$$P(X<Y) = E[P(X<Y|X)]$$

X is cont.

$$\stackrel{\downarrow}{=} \int_{-\infty}^{\infty} P(X<Y|X=x) f_X(x)\, dx$$

$$= \int_0^{\infty} P(X<Y|X=x) \cdot \lambda e^{-\lambda x}\, dx$$

$$P(\underline{X<Y} \mid \underline{X=x}) = P(x<Y)$$

X and Y are indep. So,

- drop conditioning
- turn X into x

$$P(x<Y) = e^{-\mu x}$$

b/c $Y \sim \mathrm{Exp}(\mu)$

$$P(X<Y) = \int_0^{\infty} e^{-\mu x} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-(\mu+\lambda)x}\,dx$$

$$= \lambda \cdot \left(-\frac{1}{\mu+\lambda} e^{-(\mu+\lambda)x}\right)\Bigg|_{x=0}^{x=\infty}$$

$$= \lambda\left(0 - \left(-\frac{1}{\mu+\lambda}\right)\right) = \boxed{\frac{\lambda}{\mu+\lambda}}$$

Remember, if $X \sim Exp(\lambda)$, then $E[X] = \frac{1}{\lambda}$.

The larger λ is, the smaller X is (on average).

So it makes sense that

$X \sim Exp(\lambda)$ $\quad Y \sim Exp(\mu)$ $\quad P(X<Y) = \frac{\lambda}{\mu+\lambda}$

Example 5l Let U be a uniform random variable on (0, 1), and suppose that the conditional distribution of X, given that $U = p$, is binomial with parameters n and p. Find the probability mass function of X.

$U \sim Unif(0,1)$

$$P(X=k \mid U=p) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0<p<1 \text{ and } k \in \{0,1,\dots,n\}, \\ 0 & \text{if } 0<p<1 \text{ and } k \notin \{0,1,\dots,n\}. \end{cases}$$

$P(X=k) = ?$

$$P(X=k) = E[P(X=k \mid U)]$$

U is cont.

$$\overset{\downarrow}{=} \int_{-\infty}^{\infty} P(X=k \mid U=p) f_U(p)\, dp$$

$$= \int_0^1 P(X=k \mid U=p) \cdot 1 \, dp$$

$k \in \{0, 1, \ldots, n\}$:

$$P(X=k) = \int_0^1 \binom{n}{k} p^k (1-p)^{n-k} \, dp$$

$$= \binom{n}{k} \int_0^1 p^k (1-p)^{n-k} \, dp$$

How to integrate this?

If we hadn't skipped the Beta distribution, you would know that

$$\int_0^1 x^{a-1} (1-x)^{b-1} \, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

You don't need to know this. If you need it, I'll give it to you.

This gives us

$$P(X=k) = \binom{n}{k} \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)}$$

$$= \frac{\cancel{n!}}{\cancel{k!}\,\cancel{(n-k)!}} \cdot \frac{\cancel{k!}\,\cancel{(n-k)!}}{(n+1)\cancel{!}} = \frac{1}{n+1}$$

$$\boxed{P(X=k) = \begin{cases} \frac{1}{n+1} & \text{if } k \in \{0,1,\dots,n\}, \\ 0 & \text{o.w.} \end{cases}}$$

Example 5m Suppose that X and Y are independent continuous random variables having densities f_X and f_Y, respectively. Compute $P\{X < Y\}$.

Soln 1

$$P(X<Y) = E[P(X<Y \mid Y)]$$

Y is cont.

$$\overset{\downarrow}{=} \int_{-\infty}^{\infty} P(X<Y \mid Y=y) f_Y(y)\,dy$$

b/c X and Y are indep.

$$\overset{\downarrow}{=} \int_{-\infty}^{\infty} P(X<y) f_Y(y)\,dy$$

$$= \boxed{\int_{-\infty}^{\infty} F_X(y) f_Y(y)\, dy}$$

Soln 2

$$P(X<Y) = E[P(X<Y \mid X)]$$

$$= \int_{-\infty}^{\infty} P(X<Y \mid X=x) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(x<Y) f_X(x)\, dx$$

$$= \boxed{\int_{-\infty}^{\infty} (1-F_Y(x)) f_X(x)\, dx}$$

HW: Ch. 7: 30, 36, 37, 40, 42, 52, 53, 55, 56, 58, 73, 74

38, 40, 50, 51, 53, 54, 56, 68, 69