

§§ 2.1-2.3: Axioms of probability

Intuitive axioms of probability

**Example
2a**

Joe is 80 percent certain that his missing key is in one of the two pockets of his hanging jacket, being 40 percent certain it is in the left-hand pocket and 40 percent certain it is in the right-hand pocket. If a search of the left-hand pocket does not find the key, what is the conditional probability that it is in the other pocket?

↑
From Chapter 3;
this is just a
preview

We assign probabilities to propositions
(declarative sentences with a truth value).

E.g.

L = "The key is in the left-hand pocket."

R = "The key is in the right-hand pocket."

In every problem, we have some
background information (hidden assumptions)

In this problem, it is $\neg(L \& R)$
 ↑ ↑
 not and

We denote the background information by Ω .

So in this problem, $\Omega = \neg(L \& R)$.

We are asked to compute $\underbrace{P(R|\Omega \& \neg L)}$

probability of R given $\Omega \& \neg L$

probabilities are relationships

We are given:

$$P(R \text{ or } L | \Omega) = 0.8$$

$$P(R|\Omega) = 0.4$$

$$P(L|\Omega) = 0.4$$

To go from the givens to the desired quantity, we use axioms.

Axiom I: all probabilities are between 0 and 1

Axiom II: if $B \Rightarrow A$, then $P(A|B) = 1$.
↑
logically implies

Axiom III: if $C \Rightarrow \neg(A \& B)$, then

III: If $C \rightarrow (A \& B)$, then

$$P(A \text{ or } B | C) = P(A | C) + P(B | C) \leftarrow \text{addition rule}$$

Axiom IV: if $P(A|C) > 0$, then

↓: if $P(A|C) > 0$, then

$$P(A \& B|C) = P(A|C) P(B|A \& C) \leftarrow \begin{matrix} \text{multiplication} \\ \text{rule} \end{matrix}$$

Axiom V: if $A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \dots$, then

Axiom V: if $A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \dots$, then

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } \dots | C) = \lim_{n \rightarrow \infty} P(A_n | C)$$

continuity rule

To solve this problem, we'll also need one theorem.

Thm VI $P(\neg A|C) = 1 - P(A|C)$.

Pf: $A \& \neg A$ is always false, so $C \Rightarrow \neg(A \& \neg A)$.

$\therefore$ by add. rule (Axiom III),

$$P(A \text{ or } \neg A | C) = P(A|C) + P(\neg A|C).$$

But $A \text{ or } \neg A$ is always true, so $C \Rightarrow A \text{ or } \neg A$. By Axiom II, $P(A \text{ or } \neg A | C) = 1$. $\therefore$

$$1 = P(A|C) + P(\neg A|C)$$

$$\Rightarrow P(\neg A|C) = 1 - P(A|C). \quad \square$$

Solution:

$$\Omega = \neg(R \& L)$$

$$\Omega \& R \Rightarrow \neg L$$

$$P(\neg L | \Omega \& R) = 1 \quad (\text{Axiom II})$$

$$P(R \& \neg L | \Omega) = \overset{0.4}{P(R|\Omega)} \overset{1}{P(\neg L|\Omega \& R)} \quad (\text{mult. rule})$$

$$P(\neg L | \Omega) = 1 - \overset{0.4}{P(L|\Omega)} = 0.6 \quad (\text{Thm VI})$$

$$\overset{0.4}{P(\neg L \& R | \Omega)} = \overset{0.6}{P(\neg L | \Omega)} P(R | \Omega \& \neg L) \quad (\text{mult. rule})$$

$$P(R | \Omega \& \neg L) = \frac{0.4}{0.6} = \boxed{\frac{2}{3}}$$

Ω is always given, so drop it from the notation:

$$P(A) = P(A|\Omega) \quad , \quad P(A|B) = P(A|\Omega \& B)$$

Solution looks simpler:

$$\overset{0.6}{P(\neg L)} P(R|\neg L) = P(R \& \neg L) = \overset{0.4}{P(R)} \overset{1}{P(\neg L|R)}$$

$$\therefore P(R|\neg L) = \frac{0.4}{0.6} = \boxed{\frac{2}{3}}$$

Two ingredients in a probability model:

Ω : background information
propositions that are always true in the model

P : probability function
takes two propositions, A and B ,
and returns $P(A|\Omega \& B)$

or $P(A|B)$ in shorthand

To know P , it's enough to know $P(A)$ for all A .

mult. rule : $P(A \& B) = P(B) P(A|B)$

$$\Rightarrow P(A|B) = \frac{P(A \& B)}{P(B)}$$

Formal axioms of probability

All truth assignments to R, L that make $\Omega = \neg(R \& L)$ true

Ω must be true $\rightarrow$

R	L	$R \& L$	$\neg(R \& L)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$\leftarrow \omega_1$
 $\leftarrow \omega_2$
 $\leftarrow \omega_3$ } outcomes

ω_1 : outcome where key is in right pocket

ω_2 : outcome where key is in left pocket

ω_3 : outcome where key is in neither pocket

Propositions become sets:

$R = \{\omega_1\} \leftarrow$ outcomes where R is true

$\neg R = \{\omega_2, \omega_3\} \leftarrow$ outcomes where $\neg R$ is true

$R \text{ or } L = \{\omega_1, \omega_2\} \leftarrow$ outcomes where $R \text{ or } L$ is true

In general,

$$\neg A = A^c, \quad A \text{ or } B = A \cup B, \quad A \& B = A \cap B$$

Also, $A \Rightarrow B$ means $A \subseteq B$. ($w \in A$ means A is true under outcome w)

Finally,

$\Omega = \{\omega_1, \omega_2, \omega_3\}$ $\leftarrow$ outcomes where $\neg(R \& L)$ is true

$\uparrow$ the set of all outcomes
aka (sample space)

A set of outcomes is called an event

The "description" of an event is the proposition that corresponds to it.

Axiom 1 $0 \leq P(A) \leq 1$ $\leftarrow$ an event

Axiom 2 $P(\Omega) = 1$

Axiom 3 If $A_1, A_2, \dots$ are mutually exclusive (means that $A_i \cap A_j = \emptyset$ whenever $i \neq j$), then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n) \quad \leftarrow \text{gives add. rule and cont. rule}$$

Definition If $P(B) > 0$, then $P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\leftarrow$ gives mult. rule
(of conditional probability)

(Axioms in §2.3, Defn in §3.2)

Here's our solution with the formal axioms:

$$\overset{0.6}{P(L)} P(R|L^c) = P(R \cap L^c) = \overset{0.4}{P(R)} \overset{1}{P(L^c|R)}$$

$$\therefore P(R|L^c) = \frac{0.4}{0.6} = \boxed{\frac{2}{3}}$$

HW Ch.2: 3, 6, 8 (highlights are focus problems)