

# The Principles of Probability

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# Syntax

$L$ : "extralogical signature"

set of "extralogical" symbols (constants, relations, functions)

$Var$ : uncountable set of variable symbols

logical symbols :  $\neg, \wedge, \forall, \Rightarrow$

# Terms

$\mathcal{T}$ : set of "terms"

- constant  $c \in L$  is a term
- $x \in \text{Var}$  is a term
- function  $f \in L$  and  $t_1, \dots, t_n$  terms  
 $\Rightarrow f t_1 \dots t_n$  a term

# Formulas

$\mathcal{L}$ : set of "formulas"

- $s, t \in \mathcal{T} \Rightarrow s \equiv t$  a formula
- relation  $r \in L$  and  $t_1, \dots, t_n$  terms  
 $\Rightarrow r t_1 \dots t_n$  a formula

} atomic formulas

- $\varphi \in \mathcal{L}, x \in \text{Var} \Rightarrow \neg \varphi, \forall x \varphi \in \mathcal{L}$
- $\Phi \subseteq \mathcal{L}$  countable  $\Rightarrow \bigwedge \Phi \in \mathcal{L}$

Use shorthand for other formulas

## Free variables

free  $\varphi$  : set of free variables in  $\varphi$

$$\text{free } X = \bigcup_{\varphi \in X} \text{free } \varphi$$

$L^0$  : set of "sentences" (formulas w/o free variables)

$\varphi(t/x)$  : substitute each free  $x$  w/  $t \in \mathcal{T}$

" $t$  is free for  $x$  in  $\varphi$ " : no variable in  $t$  becomes bound after substitution

## Inductive statements

an "inductive statement":

$$(X, \varphi, p) \in \mathcal{L}^{IS} = \mathbb{R}^{\mathcal{L}^0} \times \mathcal{L}^0 \times [0, 1]$$

for  $\mathcal{P} \subseteq \mathcal{L}^{IS}$ :

$$\text{ante } \mathcal{P} = \{X \in \mathcal{L}^0 \mid (X, \varphi, p) \in \mathcal{P} \text{ for some } \varphi, p\}$$

## Deductive calculus

- $\varphi \vdash \varphi$
- $X \vdash \varphi$  and  $X \subseteq X' \Rightarrow X' \vdash \varphi$
- $X \vdash \varphi$  and  $X \vdash \neg \varphi \Rightarrow X \vdash \perp$
- $X, \varphi \vdash \psi$  and  $X, \neg \varphi \vdash \psi \Rightarrow X \vdash \psi$
- $X \vdash \bigwedge \Phi \Leftrightarrow X \vdash \Theta$  for all  $\Theta \in \Phi$
- $X \vdash \forall x \varphi \Rightarrow X \vdash \varphi(t/x)$  when  $t$  is free for  $x$
- $x \notin \text{free } X$  and  $X \vdash \varphi \Rightarrow X \vdash \forall x \varphi$
- $\vdash t = t$  for all  $t \in \mathcal{T}$
- $X \vdash s = t, \varphi(t/x) \Rightarrow X \vdash \varphi(t/x)$  when  $t$  is free for  $x$

## Logical equivalence

$X \vdash Y$  :  $X \vdash \psi$  for all  $\psi \in Y$

$\varphi \equiv \psi$  :  $\varphi \vdash \psi$  and  $\psi \vdash \varphi$

$X \equiv Y$  :  $X \vdash Y$  and  $Y \vdash X$

$\varphi \equiv_x \psi$  :  $X, \varphi \vdash \psi$  and  $X, \psi \vdash \varphi$

## Deductive theories

$X$  is "consistent" if  $X \not\vdash \varphi$  for some  $\varphi \in \mathcal{L}$

$T \subseteq \mathcal{L}^\circ$  is a "(deductive) theory" if

$$T \vdash \varphi \Rightarrow \varphi \in T \text{ for all } \varphi \in \mathcal{L}^\circ$$

for  $X \subseteq \mathcal{L}^\circ$ ,

$T(X) = T_X = \{\varphi \in \mathcal{L}^\circ \mid X \vdash \varphi\}$  : "theory generated by  $X$ "  
(smallest theory containing  $X$ )

$$X \vdash \varphi \text{ iff } \varphi \in T(X)$$

## Inductive calculus

(R1) rule of logical equivalence

- $(X, \varphi, p) \in \mathcal{P}, X' \equiv X, \varphi' \equiv_x \varphi \Rightarrow (X', \varphi', p) \in \mathcal{P}$
- $(X, \varphi, p), (X, \varphi, p') \in \mathcal{P} \Rightarrow p = p'$

If  $\mathcal{P}$  satisfies (R1), can write

$$P(\varphi | X) = p \quad \text{instead of} \quad (X, \varphi, p) \in \mathcal{P}$$

shorthand:

$$P(\varphi | X, \psi) = P(\varphi | X \cup \{\psi\})$$

$$P(\varphi) = P(\varphi | \emptyset)$$

" $P(\varphi | X)$  exists" means  $(X, \varphi, p) \in \mathcal{P}$  for some  $p$

(R2) rule of logical implication

$$X \in \text{ante } P, X \vdash \varphi \Rightarrow P(\varphi | X) = 1$$

(R3) rule of material implication

$$X \in \text{ante } P, P(\varphi | X, \psi) = 1 \Rightarrow P(\psi \rightarrow \varphi | X) = 1$$

(R4) rule of deductive transitivity

- $P(\varphi | X) = 1, \varphi \vdash \psi \Rightarrow P(\psi | X) = 1$

- $X' \in \text{ante } P, X' \vdash X, P(\varphi | X) = 1 \Rightarrow P(\varphi | X') = 1$

(R5) the addition rule

Let  $X \vdash \neg(\varphi \wedge \psi)$ . Consider the eq.

$$P(\varphi \vee \psi | X) = P(\varphi | X) + P(\psi | X).$$

If two of these probabilities exist,

then so does the third, and the eq. holds.

(R6) the multiplication rule

Consider the eq.

$$P(\varphi \wedge \psi | X) = P(\varphi | X) P(\psi | X, \varphi).$$

If two of these probabilities exist and are positive,

then so does the third, and the eq. holds.

(R7) the continuity rule

If  $P(\varphi_n | X)$  exists and  $X, \varphi_n \vdash \varphi_{n+1}$  for all  $n$ , then

$$P(\bigvee_n \varphi_n | X) = \lim_{n \rightarrow \infty} P(\varphi_n | X)$$

Let  $\mathcal{P} \subseteq \bar{\mathcal{P}} \subseteq \mathcal{L}^{IS}$ . Then  $\bar{\mathcal{P}}$  is a "completion" of  $\mathcal{P}$  if

- $\bar{\mathcal{P}}$  satisfies (R1)–(R7)
- $\bar{\mathcal{P}}(\varphi|X), \bar{\mathcal{P}}(\psi|X)$  exist  $\Rightarrow \bar{\mathcal{P}}(\varphi \wedge \psi|X)$  exists
- $X, X \cup \{\varphi\} \in \text{ante } \bar{\mathcal{P}} \Rightarrow \bar{\mathcal{P}}(\varphi|X)$  exists

(R8) the rule of inductive extension

If  $\bar{\mathcal{P}}(\varphi|X) = p$  for every completion  $\bar{\mathcal{P}}$  of  $\mathcal{P}$ ,  
then  $\mathcal{P}(\varphi|X) = p$ .

(R9) the rule of deductive extension

If  $S \subseteq \mathcal{L}^0$  is nonempty and  $\mathcal{P}(\theta|X) = 1$  for all  $\theta \in S$ ,  
then  $X \cup S \in \text{ante } \mathcal{P}$  and  $\mathcal{P}(\cdot|X, S) = \mathcal{P}(\cdot|X)$

$Q \subseteq \mathcal{L}^{IS}$  is "connected" if there exists  $\hat{Q} \subseteq Q$  s.t.

- every  $\hat{X} \in \text{ante } \hat{Q}$  is countably axiomatizable over a common  $X_0 \in \text{ante } \hat{Q}$ .
- $X \in \text{ante } Q \Rightarrow X \equiv \hat{X} \cup S$  for some  $\hat{X} \in \text{ante } \hat{Q}$  and some  $S \subseteq \{\theta \in \mathcal{L}^0 \mid (\hat{X}, \theta, \perp) \in \hat{Q}\}$

In this case,  $T_0 = T(X_0)$  is unique and is called the "root" of  $Q$ .

$P \subseteq \mathcal{L}^{IS}$  is an "inductive theory" if it satisfies (R1) - (R9) and is connected.

$Q \subseteq \mathcal{L}^{IS}$  is "consistent" if it is connected and can be extended to an inductive theory.

For consistent  $Q$ ,

$\underline{P}(Q) = \underline{P}_Q =$  smallest inductive theory containing  $Q$ .

$Q \vdash (X, \varphi, p)$  means  $Q$  is consistent and  $\underline{P}_Q(\varphi | X) = p$ .

$\mathcal{C}$  is an "inductive condition" if  $\mathcal{C}$  is a set of inductive theories w/a common root  $T_0$ .

$T_0$  is called the "root" of  $\mathcal{C}$ .

$\mathcal{C}$  is "consistent" if  $\mathcal{C} \neq \emptyset$ .

For consistent  $\mathcal{C}$ ,

$\underline{P}(\mathcal{C}) = \underline{P}_{\mathcal{C}} =$  largest inductive theory in  $\bigcap \mathcal{C}$ .

$\mathcal{C} \vdash (X, \mathcal{Q}, p)$  means  $\mathcal{C}$  is consistent and  $\underline{P}_{\mathcal{C}}(\mathcal{Q} \mid X) = p$ .

## Coin Flip Example

$L = \{c_1, c_2, h, t\}$ , all constant symbols

$$\varphi_0 : h \neq t$$

$$\varphi_i : c_i \neq h \vee c_i \neq t, \quad i = 1, 2$$

$$T_0 = T(\{\varphi_0, \varphi_1, \varphi_2\})$$

$$Q = \{(T_0, c_i \neq h, \frac{2}{3}) \mid i = 1, 2\}$$

Can show that  $\underline{P}_Q(c_i \neq t \mid T_0) = \frac{1}{3}$ ,

$$\text{i.e. } Q \vdash (T_0, c_i \neq t, \frac{1}{3})$$

$\mathcal{C}$  = set of inductive theories  $P$  w/ root  $T_0$

s.t.  $Q \subseteq P$  and

$$P(c_2 \models h \mid T_0, c_1 \models h) = P(c_2 \models h \mid T_0)$$

Can show that  $\underline{P}_{\mathcal{C}}(c_1 \models c_2 \models h \mid T_0) = \frac{4}{9}$ ,

i.e.  $\mathcal{C} \vdash (T_0, c_1 \models c_2 \models h, \frac{4}{9})$

## Models

$A =$  a set

$\omega =$  a "structure" with domain  $A$

For  $\underline{s} \in L$ ,  $\underline{s}^\omega$  is an actual object/relation in  $A$

$$L^\omega = \{ \underline{s}^\omega \mid \underline{s} \in L \}$$

$$\omega = (A, L^\omega)$$

$v: \text{Var} \rightarrow A$

Extend to  $v: \mathcal{T} \rightarrow A$ , "assignment into  $\omega$ "

$\omega \models \mathcal{Q}[v]$  means  $\mathcal{Q}$  is true in  $\omega$  when free variables are assigned according to  $v$   
("strict satisfiability")

$\Omega =$  set of structures

"model" :  $\mathcal{P} = (\Omega, \Sigma, \mathbb{P})$

$\underline{v} = \langle v_\omega \mid \omega \in \Omega \rangle$  : "assignment into  $\mathcal{P}$ "

For  $\varphi \in \mathcal{L}$ :

$$\varphi[\underline{v}]_\Omega = \{ \omega \in \Omega \mid \omega \models \varphi[v_\omega] \}$$

$$\mathcal{P} \models \varphi[\underline{v}] \text{ iff } \varphi[\underline{v}]_\Omega \in \overline{\Sigma} \text{ and } \overline{\mathbb{P}} \varphi[\underline{v}]_\Omega = 1$$

For  $\varphi \in \mathcal{L}^0$ :

$$\varphi_\Omega = \{ \omega \in \Omega \mid \omega \models \varphi \}$$

$$\mathcal{P} \models \varphi \text{ iff } \varphi_\Omega \in \overline{\Sigma} \text{ and } \overline{\mathbb{P}} \varphi_\Omega = 1$$

$X \subseteq \mathcal{L}$  is "satisfiable" if, for some  $\mathcal{P}, \underline{v}$ ,

$$\mathcal{P} \models \psi[\underline{v}] \text{ for all } \psi \in X$$

$X$  is consistent iff  $X$  is satisfiable

$X$  is satisfiable iff every countable subset of  $X$  is satisfiable

$X \models \varphi$  means: for all  $\mathcal{P}, \underline{v}$ ,

if  $\mathcal{P} \models \psi[\underline{v}]$  for all  $\psi \in X$ ,

then  $\mathcal{P} \models \varphi[\underline{v}]$

$X \vdash \varphi$  iff  $X \models \varphi$

$\mathcal{P} \models (X, \varphi, p) \in \mathcal{L}^{IS}$  if, for some  $Y, \psi$ ,

- $X \equiv Y \cup \{\psi\}$

- $\mathcal{P} \models Y$

- $\frac{\overline{\mathcal{P}} \varphi_{\Omega} \wedge \psi_{\Omega}}{\overline{\mathcal{P}} \psi_{\Omega}} = p$

$\mathcal{P} \models Q \subseteq \mathcal{L}^{IS}$  if  $\mathcal{P} \models (X, \varphi, p)$  for all  $(X, \varphi, p) \in Q$

$Q$  is "satisfiable" if  $\mathcal{P} \models Q$  for some  $\mathcal{P}$

$Q$  is consistent iff  $Q$  is connected and satisfiable

$\mathcal{Q}$ : consistent w/ root  $T_0$

$T_{\mathcal{Q}} = \{ \theta \in \mathcal{L}^0 \mid \text{for all } \mathcal{P},$   
if  $\mathcal{P} \models \mathcal{Q}$  and  $\mathcal{P} \models T_0$ , then  $\mathcal{P} \models \theta \}$

$T_{\mathcal{Q}}$  is a consistent deductive theory w/  $T_0 \subseteq T_{\mathcal{Q}}$

$\mathcal{Q} \models (X, \mathcal{L}, p)$  means:

- $\mathcal{Q}$  is consistent
- $X$  is countably axiomatizable over some deductive theory  $T$  with  $T_0 \subseteq T \subseteq T_{\mathcal{Q}}$
- for all  $\mathcal{P}$ , if  $\mathcal{P} \models \mathcal{Q}$ , then  $\mathcal{P} \models (X, \mathcal{L}, p)$

$\mathcal{Q} \vdash (X, \mathcal{L}, p)$  iff  $\mathcal{Q} \models (X, \mathcal{L}, p)$

$\mathcal{L}$  : an inductive condition

$\mathcal{P} \models \mathcal{L}$  if  $\mathcal{P} \models P$  for some  $P \in \mathcal{L}$

$\mathcal{L} \models (X, \mathcal{Q}, p)$  defined as above

$\mathcal{L}$  is consistent iff  $\mathcal{L}$  is satisfiable

$\mathcal{L} \vdash (X, \mathcal{Q}, p)$  iff  $\mathcal{L} \models (X, \mathcal{Q}, p)$

## Coin Flip Example (again)

$$A = \{0, 1\}, \quad \Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}, \quad \Sigma = \mathcal{P}\Omega$$

$$\mathcal{P} = (\Omega, \Sigma, \mathbb{P})$$

$\omega$	$h^\omega$	$t^\omega$	$c_1^\omega$	$c_2^\omega$	$\mathbb{P}\{\omega\}$
$\omega_0$	1	0	0	0	1/9
$\omega_1$	1	0	0	1	2/9
$\omega_2$	1	0	1	0	2/9
$\omega_3$	1	0	1	1	4/9

Can show that  $\mathcal{P} \neq \mathcal{Q}$ . To show  $\mathcal{P} \neq \mathcal{C}$ , let

$$P = \{(X, \mathcal{L}, p) \in \mathcal{L}^{IS} \mid \mathcal{P} \neq (X, \mathcal{L}, p) \text{ and } X \vdash T_0\}$$

Can show that  $P$  is an inductive theory and  $P \in \mathcal{C}$ .

## Embedding theorems

$(S, \Gamma, \nu)$ , probability space

$\langle (R_i, \Gamma_i) \mid i \in I \rangle$ , measurable spaces

$X_i: \Omega \rightarrow R_i$ , random variables

$X = \langle X_i \mid i \in I \rangle$

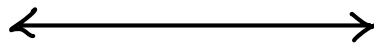
$(S, \Gamma, \nu, X)$  is a "measure-theoretic probability model"

one-to-one proper embedding

$$(S, \Gamma, \nu, X) \longleftrightarrow \mathbb{P} = (\Omega, \Sigma, \mathbb{P})$$

outcomes

$$x \in S$$

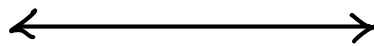


structures

$$\omega \in \Omega$$

events

$$U \in \Gamma$$

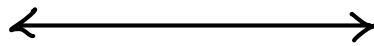


sentences

$$\varphi \in \mathcal{L}^0$$

set membership

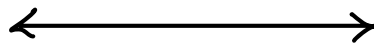
$$x \in U$$



strict satisfiability

$$\omega \models \varphi$$

random variables



constant symbols