The Principles of Probability

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UCF Probabity Seminar, April 27, 2026

Syntax

L: "extralogical signature"

set of "extralogical" symbols (constants, relations, functions)

Var: uncountable set of variable symbols

logical symbols : $\neg$, $\wedge$, $\forall$, $=$

Terms

$\mathcal{T}$: set of "terms"

- constant $c \in L$ is a term
- $x \in Var$ is a term
- function $f \in L$ and $t_1, \ldots, t_n$ terms
  $\Longrightarrow f t_1 \cdots t_n$ a term

Formulas

$\mathcal{L}$: set of "formulas"

$$\left.\begin{array}{l} \bullet\ s, t \in \mathcal{T} \Rightarrow s \doteq t \text{ a formula} \\[1em] \bullet\ \text{relation } r \in L \text{ and } t_1, \ldots, t_n \text{ terms} \\ \qquad \Rightarrow r t_1 \cdots t_n \text{ a formula} \end{array}\right\} \text{atomic formulas}$$

- $\varphi \in \mathcal{L}$, $x \in Var \Rightarrow \neg\varphi, \forall x \varphi \in \mathcal{L}$
- $\Phi \subseteq \mathcal{L}$ countable $\Rightarrow \bigwedge \Phi \in \mathcal{L}$

Use shorthand for other formulas

Free variables

free φ : set of free variables in φ

free $X = \bigcup_{\varphi \in X}$ free φ

$\mathcal{L}^0$: set of "sentences" (formulas w/o free variables)

$\varphi(t/x)$: substitute each free x w/ $t \in \mathcal{T}$

"t is free for x in φ" : no variable in t becomes bound after substitution

Inductive statements

an "inductive statement":

$$(X, \varphi, p) \in \mathcal{L}^{IS} = \not{P}\mathcal{L}^0 \times \mathcal{L}^0 \times [0,1]$$

for $P \subseteq \mathcal{L}^{IS}$:

$$\text{ante } P = \{ X \subseteq \mathcal{L}^0 \mid (X, \varphi, p) \in P \text{ for some } \varphi, p \}$$

Deductive calculus

- $\varphi \vdash \varphi$
- $X \vdash \varphi$ and $X \subseteq X' \Rightarrow X' \vdash \varphi$
- $X \vdash \varphi$ and $X \vdash \neg\varphi \Rightarrow X \vdash \psi$
- $X, \varphi \vdash \psi$ and $X, \neg\varphi \vdash \psi \Rightarrow X \vdash \psi$
- $X \vdash \bigwedge \Phi \Leftrightarrow X \vdash \theta$ for all $\theta \in \Phi$
- $X \vdash \forall x \varphi \Rightarrow X \vdash \varphi(t/x)$ when t is free for x
- $x \notin \mathrm{free}\, X$ and $X \vdash \varphi \Rightarrow X \vdash \forall x \varphi$
- $\vdash t \doteq t$ for all $t \in \mathcal{T}$
- $X \vdash s \doteq t, \varphi(t/x) \Rightarrow X \vdash \varphi(t/x)$ when t is free for x

Logical equivalence

$X \vdash Y \quad : \quad X \vdash \psi$ for all $\psi \in Y$

$\varphi \equiv \psi \quad : \quad \varphi \vdash \psi$ and $\psi \vdash \varphi$

$X \equiv Y \quad : \quad X \vdash Y$ and $Y \vdash X$

$\varphi \equiv_X \psi \quad : \quad X, \varphi \vdash \psi$ and $X, \psi \vdash \varphi$

Deductive theories

X is "consistent" if $X \nvdash \varphi$ for some $\varphi \in \mathcal{L}$

$T \subseteq \mathcal{L}^\circ$ is a "(deductive) theory" if

$$T \vdash \varphi \Rightarrow \varphi \in T \text{ for all } \varphi \in \mathcal{L}^\circ$$

for $X \subseteq \mathcal{L}^\circ$,

$T(X) = T_X = \{\varphi \in \mathcal{L}^\circ \mid X \vdash \varphi\}$: "theory generated by X"
(smallest theory containing X)

$X \vdash \varphi$ iff $\varphi \in T(X)$

Inductive calculus

(R1) rule of logical equivalence

- $(X, \varphi, p) \in P$, $X' \equiv X$, $\varphi' \equiv_X \varphi \Rightarrow (X', \varphi', p) \in P$
- $(X, \varphi, p), (X, \varphi, p') \in P \Rightarrow p = p'$

If P satisfies (R1), can write

$$P(\varphi \mid X) = p \quad \text{instead of} \quad (X, \varphi, p) \in P$$

shorthand:

$$P(\varphi \mid X, \psi) = P(\varphi \mid X \cup \{\psi\})$$

$$P(\varphi) = P(\varphi \mid \emptyset)$$

"$P(\varphi \mid X)$ exists" means $(X, \varphi, p) \in P$ for some p

(R2) rule of logical implication

$$X \in \mathrm{ante}\, P,\quad X \vdash \varphi \;\Longrightarrow\; P(\varphi \mid X) = 1$$

(R3) rule of material implication

$$X \in \mathrm{ante}\, P,\; P(\varphi \mid X, \psi) = 1 \;\Longrightarrow\; P(\psi \to \varphi \mid X) = 1$$

(R4) rule of deductive transitivity

- $P(\varphi \mid X) = 1,\quad \varphi \vdash \psi \;\Longrightarrow\; P(\psi \mid X) = 1$
- $X' \in \mathrm{ante}\, P,\; X' \vdash X,\; P(\varphi \mid X) = 1 \;\Longrightarrow\; P(\varphi \mid X') = 1$

(R5) the addition rule

Let $X \vdash \neg(\varphi \wedge \psi)$. Consider the eq.

$$P(\varphi \vee \psi \mid X) = P(\varphi \mid X) + P(\psi \mid X).$$

If two of these probabilities exist, then so does the third, and the eq. holds.

(R6) the multiplication rule

Consider the eq.

$$P(\varphi \wedge \psi \mid X) = P(\varphi \mid X)\, P(\psi \mid X, \varphi).$$

If two of these probabilities exist and are positive, then so does the third, and the eq. holds.

(R7) the continuity rule

If $P(\varphi_n \mid X)$ exists and $X, \varphi_n \vdash \varphi_{n+1}$ for all n, then

$$P(\textstyle\bigvee_n \varphi_n \mid X) = \lim_{n \to \infty} P(\varphi_n \mid X)$$

Let $P \subseteq \overline{P} \subseteq \mathcal{L}^{IS}$. Then $\overline{P}$ is a "completion" of P if

- $\overline{P}$ satisfies (R1)–(R7)
- $\overline{P}(\varphi | X), \overline{P}(\psi | X)$ exist $\Rightarrow$ $\overline{P}(\varphi \wedge \psi | X)$ exists
- $X, X \cup \{\varphi\} \in \operatorname{ante} \overline{P} \Rightarrow \overline{P}(\varphi | X)$ exists

(R8) the rule of inductive extension

If $\overline{P}(\varphi | X) = p$ for every completion $\overline{P}$ of P,

then $P(\varphi | X) = p$.

(R9) the rule of deductive extension

If $S \subseteq \mathcal{L}^0$ is nonempty and $P(\theta | X) = 1$ for all $\theta \in S$,

then $X \cup S \in \operatorname{ante} P$ and $P(\cdot | X, S) = P(\cdot | X)$

$Q \subseteq \mathcal{L}^{IS}$ is "connected" if there exists $\hat{Q} \subseteq Q$ s.t.

- every $\hat{X} \in \text{ante}\,\hat{Q}$ is countably axiomatizable over a common $X_0 \in \text{ante}\,\hat{Q}$.
- $X \in \text{ante}\,Q \Rightarrow X \equiv \hat{X} \cup S$ for some $\hat{X} \in \text{ante}\,\hat{Q}$ and some $S \subseteq \{\theta \in \mathcal{L}^0 \mid (\hat{X}, \theta, 1) \in \hat{Q}\}$

In this case, $T_0 = T(X_0)$ is unique and is called the "root" of Q.

$P \subseteq \mathcal{L}^{IS}$ is an "inductive theory" if it satisfies (R1) - (R9) and is connected.

$Q \subseteq \mathcal{L}^{IS}$ is "consistent" if it is connected and can be extended to an inductive theory.

For consistent Q,

$\underline{P}(Q) = \underline{P}_Q$ = smallest inductive theory containing Q.

$Q \vdash (X, \varphi, p)$ means Q is consistent and $\underline{P}_Q(\varphi | X) = p$.

$\mathcal{C}$ is an "inductive condition" if $\mathcal{C}$ is a set of inductive theories w/a common root T_0.

T_0 is called the "root" of $\mathcal{C}$.

$\mathcal{C}$ is "consistent" if $\mathcal{C} \neq \phi$.

For consistent $\mathcal{C}$,

$$\underline{P}(\mathcal{C}) = \underline{P}_{\mathcal{C}} = \text{largest inductive theory in } \bigcap \mathcal{C}.$$

$\mathcal{C} \vdash (X, \varphi, p)$ means $\mathcal{C}$ is consistent and $\underline{P}_{\mathcal{C}}(\varphi | X) = p$.

Coin Flip Example

$L = \{c_1, c_2, h, t\}$, all constant symbols

$\varphi_0 : \; h \not\doteq t$

$\varphi_i : \; c_i \doteq h \vee c_i \doteq t \; , \quad i = 1, 2$

$T_0 = \mathcal{T}(\{\varphi_0, \varphi_1, \varphi_2\})$

$Q = \{(T_0, c_i \doteq h, \frac{2}{3}) \mid i = 1, 2\}$

Can show that $\underline{P}_Q(c_1 \doteq t \mid T_0) = \frac{1}{3}$,

i.e. $Q \vdash (T_0, c_1 \doteq t, \frac{1}{3})$

$\mathcal{C}$ = set of inductive theories P w/ root T_0

s.t. $Q \subseteq P$ and

$$P(c_2 \eqcirc h \mid T_0, c_1 \eqcirc h) = P(c_2 \eqcirc h \mid T_0)$$

Can show that $\quad \underline{P}_{\mathcal{C}}(c_1 \eqcirc c_2 \eqcirc h \mid T_0) = \frac{4}{9}$,

i.e. $\quad \mathcal{C} \vdash (T_0, c_1 \eqcirc c_2 \eqcirc h, \frac{4}{9})$

Models

A = a set

ω = a "structure" with domain A

For $\underline{s} \in L$, $\underline{s}^{\omega}$ is an actual object/relation in A

$L^{\omega} = \{ \underline{s}^{\omega} \mid \underline{s} \in L \}$

$\omega = (A, L^{\omega})$

$v : Var \to A$

Extend to $v : \mathcal{T} \to A$, "assignment into ω"

$\omega \models \varphi[v]$ means φ is true in ω when free variables are assigned according to v ("strict satisfiability")

Ω = set of structures

"model" : $\mathcal{P} = (\Omega, \Sigma, \mathbb{P})$

$\underline{v} = \langle v_\omega \mid \omega \in \Omega \rangle$: "assignment into $\mathcal{P}$"

For $\varphi \in \mathcal{L}$:

$$\varphi[\underline{v}]_\Omega = \{\omega \in \Omega \mid \omega \models \varphi[v_\omega]\}$$

$$\mathcal{P} \models \varphi[\underline{v}] \text{ iff } \varphi[\underline{v}]_\Omega \in \overline{\Sigma} \text{ and } \overline{\mathbb{P}}\varphi[\underline{v}]_\Omega = 1$$

For $\varphi \in \mathcal{L}^0$:

$$\varphi_\Omega = \{\omega \in \Omega \mid \omega \models \varphi\}$$

$$\mathcal{P} \models \varphi \text{ iff } \varphi_\Omega \in \overline{\Sigma} \text{ and } \overline{\mathbb{P}}\varphi_\Omega = 1$$

$X \subseteq \mathcal{L}$ is "satisfiable" if, for some $\mathfrak{P}$, $\underline{v}$,

$\mathfrak{P} \models \psi[\underline{v}]$ for all $\psi \in X$

$\boxed{X \text{ is consistent iff } X \text{ is satisfiable}}$

$\boxed{\begin{array}{l} X \text{ is satisfiable iff every countable subset of } X \\ \qquad\qquad\qquad\qquad \text{is satisfiable} \end{array}}$

$X \models \varphi$ means: for all $\mathfrak{P}$, $\underline{v}$,

if $\mathfrak{P} \models \psi[\underline{v}]$ for all $\psi \in X$,

then $\mathfrak{P} \models \varphi[\underline{v}]$

$\boxed{X \vdash \varphi \text{ iff } X \models \varphi}$

$\mathbb{P} \vDash (X, \varphi, p) \in \mathcal{L}^{IS}$ if, for some Y, ψ,

- $X \equiv Y \cup \{\psi\}$
- $\mathbb{P} \vDash Y$
- $\dfrac{\overline{\mathbb{P}}\, \varphi_\Omega \cap \psi_\Omega}{\overline{\mathbb{P}}\, \psi_\Omega} = p$

$\mathbb{P} \vDash Q \subseteq \mathcal{L}^{IS}$ if $\mathbb{P} \vDash (X, \varphi, p)$ for all $(X, \varphi, p) \in Q$

Q is "satisfiable" if $\mathbb{P} \vDash Q$ for some $\mathbb{P}$

$\boxed{Q \text{ is consistent iff } Q \text{ is connected and satisfiable}}$

Q : consistent w/ root T_0

$T_Q = \{ \theta \in \mathcal{L}^0 \mid$ for all $\mathbb{P}$, if $\mathbb{P} \models Q$ and $\mathbb{P} \models T_0$, then $\mathbb{P} \models \theta \}$

T_Q is a consistent deductive theory w/ $T_0 \subseteq T_Q$

$Q \models (X, \varphi, p)$ means:

- Q is consistent
- X is countably axiomatizable over some deductive theory T with $T_0 \subseteq T \subseteq T_Q$
- for all $\mathbb{P}$, if $\mathbb{P} \models Q$, then $\mathbb{P} \models (X, \varphi, p)$

$$\boxed{Q \vdash (X, \varphi, p) \quad \text{iff} \quad Q \models (X, \varphi, p)}$$

$\mathcal{C}$: an inductive condition

$\mathbb{P} \models \mathcal{C}$ if $\mathbb{P} \models P$ for some $P \in \mathcal{C}$

$\mathcal{C} \models (X, \varphi, p)$ defined as above

$\boxed{\mathcal{C} \text{ is consistent iff } \mathcal{C} \text{ is satisfiable}}$

$\boxed{\mathcal{C} \vdash (X, \varphi, p) \text{ iff } \mathcal{C} \models (X, \varphi, p)}$

Coin Flip Example (again)

$A = \{0,1\}$, $\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}$, $\Sigma = 2^{\Omega}$

$\mathcal{P} = (\Omega, \Sigma, \mathbb{P})$

ω	h^ω	t^ω	c_1^ω	c_2^ω	$\mathbb{P}\{\omega\}$
ω_0	1	0	0	0	1/9
ω_1	1	0	0	1	2/9
ω_2	1	0	1	0	2/9
ω_3	1	0	1	1	4/9

Can show that $\mathcal{P} \vDash Q$. To show $\mathcal{P} \vDash \mathcal{C}$, let

$$P = \{(X, \varphi, p) \in \mathcal{L}^{IS} \mid \mathcal{P} \vDash (X, \varphi, p) \text{ and } X \vdash T_0\}$$

Can show that P is an inductive theory and $P \in \mathcal{C}$.

Embedding theorems

(S, Γ, ν), probability space

$\langle (R_i, \Gamma_i) \mid i \in I \rangle$, measurable spaces

$X_i : \Omega \to R_i$, random variables

$X = \langle X_i \mid i \in I \rangle$

(S, Γ, ν, X) is a "measure-theoretic probability model" one-to-one proper embedding

(S, Γ, ν, X)	$\longleftrightarrow$	$\mathfrak{P} = (\Omega, \Sigma, \mathbb{P})$
outcomes	$\longleftrightarrow$	structures
$x \in S$		$\omega \in \Omega$
events	$\longleftrightarrow$	sentences
$U \in \Gamma$		$\varphi \in \mathcal{L}^{\circ}$
set membership	$\longleftrightarrow$	strict satisfiability
$x \in U$		$\omega \Vdash \varphi$
random variables	$\longleftrightarrow$	constant symbols