§§2.1–2.3: Axioms of probability

Intuitive axioms of probability

Example 2a Joe is 80 percent certain that his missing key is in one of the two pockets of his hanging jacket, being 40 percent certain it is in the left-hand pocket and 40 percent certain it is in the right-hand pocket. If a search of the left-hand pocket does not find the key, what is the conditional probability that it is in the other pocket?

From Chapter 3; this is just a preview

We assign probabities to <u>propositions</u> (declarative sentences with a truth value).

E.g.

L = "The key is in the left-hand pocket."

R = "The key is in the right-hand pocket."

In every problem, we have some <u>background information</u> (hidden assumptions)

In this problem, it is $\neg(L \& R)$

not and

We denote the background information by Ω.

So in this problem, $\Omega = \neg(L \& R)$.

We are asked to compute $\underbrace{P(R \mid \Omega \,\&\, \neg L)}_{\text{probability of } R \text{ given } \Omega \& \neg L}$

probabilities are relationships

We are given:

$$P(R \text{ or } L \mid \Omega) = 0.8$$
$$P(R \mid \Omega) = 0.4$$
$$P(L \mid \Omega) = 0.4$$

To go from the givens to the desired quantity, we use axioms.

Axiom I: all probabilities are between 0 and 1

Axiom II: if $B \Rightarrow A$, then $P(A \mid B) = 1$.

logically implies

Axiom III: if $C \Rightarrow \neg(A \& B)$, then

$$P(A \text{ or } B \mid C) = P(A \mid C) + P(B \mid C).$$

addition rule

Axiom IV: if $P(A \mid C) > 0$, then

$$P(A \& B \mid C) = P(A \mid C)\, P(B \mid A \& C)$$

multiplication rule

Axiom V: if $A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \cdots$, then

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } \cdots) = \lim_{n \to \infty} P(A_n)$$

continuity rule

To solve this problem, we'll also need one thm.

Thm VI $\quad P(\neg A \mid C) = 1 - P(A \mid C)$.

Pf: $A \,\&\, \neg A$ is always false, so $C \Rightarrow \neg(A \,\&\, \neg A)$.

$\therefore$ by add. rule (Axiom III),

$$P(A \text{ or } \neg A \mid C) = P(A \mid C) + P(\neg A \mid C).$$

But A or $\neg A$ is always true, so $C \Rightarrow A$ or $\neg A$. By Axiom II, $P(A \text{ or } \neg A \mid C) = 1$. $\therefore$

$$1 = P(A \mid C) + P(\neg A \mid C)$$

$$\Rightarrow P(\neg A \mid C) = 1 - P(A \mid C). \square$$

Solution:

$$\Omega = \neg(R \,\&\, L)$$

$$\Omega \,\&\, R \Rightarrow \neg L$$

$$P(\neg L \mid \Omega \,\&\, R) = 1 \qquad \text{(Axiom II)}$$

$$P(R \,\&\, \neg L \mid \Omega) = \overset{0.4}{\cancel{P(R \mid \Omega)}}\, \overset{1}{\cancel{P(\neg L \mid \Omega \,\&\, R)}} \qquad \text{(mult. rule)}$$

$$P(\neg L \mid \Omega) = 1 - \overset{0.4}{\cancel{P(L \mid \Omega)}} = 0.6 \qquad \text{(Thm VI)}$$

$$\underset{0.4}{\cancel{P(\neg L \,\&\, R \mid \Omega)}} = \underset{0.6}{\cancel{P(\neg L \mid \Omega)}}\, P(R \mid \Omega \,\&\, \neg L) \qquad \text{(mult. rule)}$$

$$P(R \mid \Omega \,\&\, \neg L) = \frac{0.4}{0.6} = \boxed{\frac{2}{3}}$$

Ω is always given, so drop it from the notation:

$$P(A) = P(A \mid \Omega), \quad P(A \mid B) = P(A \mid \Omega \,\&\, B)$$

Solution looks simpler:

$$\cancel{P(\neg L)}^{\,0.6}\, P(R \mid \neg L) = P(R \,\&\, \neg L) = \cancel{P(R)}^{\,0.4}\, \cancel{P(\neg L \mid R)}^{\,1}$$

$$\therefore\ P(R \mid \neg L) = \frac{0.4}{0.6} = \boxed{\frac{2}{3}}$$

Two ingredients in a probability model:

Ω: background information
propositions that are always true in the model

P: probability function
takes two propositions, A and B,
and returns $P(A \mid \Omega \,\&\, B)$
or $P(A \mid B)$ in shorthand

To know P, it's enough to know $P(A)$ for all A.

mult. rule: $P(A \,\&\, B) = P(B)\, P(A \mid B)$

$$\Rightarrow P(A \mid B) = \frac{P(A \,\&\, B)}{P(B)}$$

Formal axioms of probability

All truth assignments to R, L that make $\Omega = \neg(R \& L)$ true

Ω must be true

R	L	R&L	¬(R&L) Ω	
~~T~~	~~T~~	~~T~~	~~F~~	
T	F	F	T	$\leftarrow \omega_1$
F	T	F	T	$\leftarrow \omega_2$
F	F	F	T	$\leftarrow \omega_3$

$\omega_1, \omega_2, \omega_3$: outcomes

ω_1: outcome where key is in right pocket

ω_2: outcome where key is in left pocket

ω_3: outcome where key is in neither pocket

Propositions become sets:

$R = \{\omega_1\}$ ← outcomes where R is true

$\neg R = \{\omega_2, \omega_3\}$ ← outcomes where $\neg R$ is true

$R \text{ or } L = \{\omega_1, \omega_2\}$ ← outcomes where R or L is true

In general,

$\neg A = A^c$, $A \text{ or } B = A \cup B$, $A \& B = A \cap B$

Also, $A \Rightarrow B$ means $A \subseteq B$. ($\omega \in A$ means A is true under outcome ω)

Finally,

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ ← outcomes where $\neg(R \,\&\, L)$ is true

the set of all outcomes
aka (sample space)

A set of outcomes is called an event

The "description" of an event is the proposition that corresponds to it.

Axiom 1 $\quad 0 \leq P(A) \leq 1$ an event

Axiom 2 $\quad P(\Omega) = 1$

Axiom 3 If $A_1, A_2, \ldots$ are mutually exclusive (means that $A_i \cap A_j = \emptyset$ whenever $i \neq j$), then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$ ← gives add. rule and cont. rule

Definition (of conditional probability) If $P(B) > 0$, then $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ ← gives mult. rule

(Axioms in §2.3, Defn in §3.2)

Here's our solution with the formal axioms:

$$\overset{0.6}{\cancel{P(L^c)}}\,P(R\mid L^c) = P(R\cap L^c) = \overset{0.4}{\cancel{P(R)}}\ \overset{1}{\cancel{P(L^c\mid R)}}$$

$$\therefore\ P(R\mid L^c) = \frac{0.4}{0.6} = \boxed{\frac{2}{3}}$$

<u>HW</u> Ch. 2: 3, 6, 8 (highlights are focus problems)