final review Sections: 6.1-6.5, 7.1, 7.4

(all sections: 1.1-1.5, 1.7-1.9, 2.1-2.3,
3.1-3.2, 4.1-4.6, 5.1-5.3,
above)

6.1

inner product / dot product : $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$

(in $\mathbb{R}^n$)

length/norm : $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$

unit vector : $\|\vec{u}\| = 1$

$\vec{v} \neq \vec{0} \Rightarrow \underbrace{\frac{1}{\|\vec{v}\|}}_{\text{"normalize" } \vec{v}}\vec{v}$ is a unit vector

distance: $\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

orthogonal / perpendicular : $\vec{u}\cdot\vec{v} = 0$

$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$

Pythagorean thm: If $\vec{u},\vec{v}$ orthogonal,
then $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

a subspace

$\vec{z}$ is orthogonal to W if $\vec{z}\cdot\vec{x} = 0 \ \forall \vec{x}\in W$

$W^\perp = \{\vec{z} : \vec{z}$ is orth. to $W\}$ "W perp" "orth. complement of W"

$\text{Nul } A = (\text{Row } A)^\perp$, $\text{Nul}(A^T) = (\text{Col } A)^\perp$

6.2 orthogonal set : $\{\vec{u}_1, \dots, \vec{u}_p\} \subset \mathbb{R}^n$

$$\vec{u}_i \cdot \vec{u}_j = 0 \quad \forall i \neq j$$

An orthog. set of nonzero vects. is lin. indep.

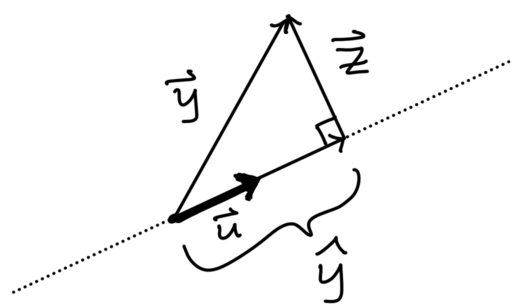
orthog. basis = orthog. set that's also a basis

$$\vec{y} \in \text{Span}\,\underbrace{\{\vec{u}_1, \dots, \vec{u}_p\}}_{\text{orthog. basis}}$$

$$\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

$$\hat{y} = \text{proj}_{\vec{u}}\,\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

"projection of $\vec{y}$ onto $\vec{u}$"



$\vec{y} = \hat{y} + \vec{z}$ orthog. to $\vec{u}$

distance from $\vec{y}$ to $\underbrace{\text{Span}\{\vec{u}\}}_{\text{a line}}$: $\|\vec{z}\| = \|\vec{y} - \hat{y}\|$

orthonormal set/basis = orthog. set/basis of unit vectors

$\overset{m \times n}{U}$ has ON cols $\iff$ $U^T U = I$

$\Downarrow$

$(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$ (preserves length & orthogonality)

orthogonal matrix = square matrix with ON cols
(in this case $U^T = U^{-1}$)

6.3

$W \subset \mathbb{R}^n$ subsp., $\{\vec{u}_1, \dots, \vec{u}_p\}$ orthog. basis, $\vec{y} \in \mathbb{R}^n$

$$\underbrace{\hat{y} = \text{proj}_W \vec{y}}_{\text{orthogonal projection of } \vec{y} \text{ onto } W} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p$$

$$\vec{y} = \underset{\text{in } W}{\hat{y}} + \underset{\text{in } W^\perp}{\vec{z}}$$ ← orthogonal decomposition (unique)

distance from $\vec{y}$ to W : $\|\vec{z}\| = \|\vec{y} - \hat{y}\|$

$\hat{y}$ is the closest pt. in W to $\vec{y}$:
if $\vec{v} \in W$ and $\vec{v} \neq \hat{y}$, then $\|\vec{v} - \vec{y}\| > \|\hat{y} - \vec{y}\|$

Let $U = (\vec{u}_1 \cdots \vec{u}_p)$. Then

$$\text{proj}_W \vec{y} = \underbrace{UU^T}_{\text{projection operator}} \vec{y}$$

6.4

$W \subset \mathbb{R}^n$ subsp.

$\{\vec{x}_1, \ldots, \vec{x}_p\}$ basis $\xrightarrow{\text{G-S}}$ $\{\underbrace{\vec{v}_1, \ldots, \vec{v}_p}\}$ orthog. basis

normalize afterward to get an orthonormal basis

G-S:

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}$$

can replace $\vec{v}_k$ by $c\vec{v}_k$ $(c \neq 0)$ after each step

- $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{Span}\{\vec{x}_1, \ldots, \vec{x}_k\}$ for all k

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QR factorization

$A = Q\,R$

A: $m \times n$, lin. indep. cols

Q: $m \times n$, ON cols

R: $n \times n$, upper triangular, positive numbers on diagonal

$A = (\vec{a}_1 \cdots \vec{a}_n)$

$\{\vec{a}_1, \dots, \vec{a}_n\}$ basis for Col A $\xrightarrow{\text{G-S}}$ $\{\vec{u}_1, \dots, \vec{u}_n\}$ ON basis for Col A

$Q = (\vec{u}_1 \cdots \vec{u}_n)$, $\quad R = Q^T A$

6.5

$\hat{x}$: a least-squares soln to $A\vec{x} = \vec{b}$

$\hat{x}$ minimizes $\|\vec{b} - A\vec{x}\|$

least squares error: $\|\vec{b} - A\hat{x}\|$

$\hat{b} = \text{proj}_{\text{Col}\,A}\,\vec{b}$ ← easy to compute if A has orthogonal cols

Solve $A\hat{x} = \hat{b}$

If $\hat{b}$ hard to compute, solve

$$A^T A \vec{x} = A^T \vec{b}$$ ← the normal eqns for $A\vec{x} = \vec{b}$

$\hat{x}$ is a least-sq. soln iff $\hat{x}$ solves the normal eqns

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Can also use QR factorization:

$$A = QR \Rightarrow \hat{x} = R^{-1} Q^T \vec{b}$$

(solve $R\hat{x} = Q^T \vec{b}$ to find $\hat{x}$)

7.1

symmetric matrix: $A^T = A$

A symmetric $\Rightarrow$ eigvects. corr. to different eigvals. are orthogonal

orthogonally diagonalizable:

$$A = PDP^{-1} = PDP^T \qquad (P^T = P^{-1})$$

A: $n \times n$

P: $n \times n$ orthogonal (means it has ON cols)

D: $n \times n$ diagonal

A is orthog. diag. $\Longleftrightarrow$ A is symmetric

Spectral decomposition

A symmetric, $\{\vec{u}_1, \dots, \vec{u}_n\}$ ON eigvects. corr. to $\lambda_1, \dots, \lambda_n$

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_k \underbrace{\vec{u}_k \vec{u}_k^T}_{\substack{n\times n \text{ matrix,} \\ \text{projects onto } \vec{u}_k}} + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

7.4

Singular value decomposition (SVD)

A: any matrix ($m \times n$)

eigvals of A^TA:

r = Rank A

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$$

ON eigvects of A^TA:

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n$$

($\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is an orthog. basis for Col A)

singular values of A:

$$\sigma_k = \sqrt{\lambda_k} = \|A\vec{v}_k\|$$

$r \times r$ $\qquad\qquad\qquad\qquad\qquad$ $m \times n$

$$D = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = (\vec{v}_1 \cdots \vec{v}_n)$$

$n \times n$

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1, \quad \ldots, \quad \vec{u}_r = \frac{1}{\sigma_r} A \vec{v}_r$$

Extend $\{\vec{u}_1, \ldots, \vec{u}_r\}$ to

$\{\vec{u}_1, \ldots, \vec{u}_m\} \leftarrow$ ON basis for $\mathbb{R}^m$

$$U = (\vec{u}_1 \cdots \vec{u}_m)$$

$m \times m$

$$A = U \Sigma V^T$$

singular value decomposition