7.1 Diagonalization of symmetric matrices

A <u>symmetric</u> matrix is one with $A^T = A$.

($A^T = A \Rightarrow A$ is a square matrix, so a symmetric matrix must be square)

Expl 2

Diagonalize $A = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$

match

match

Because of these matches, A is symmetric

$$\det(A - \lambda I) = -\lambda^3 + 17\lambda^2 - 90\lambda + 144$$
$$= -(\lambda - 8)(\lambda - 6)(\lambda - 3) = 0$$

$\lambda_1 = 8$: $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 6$: $\vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$\lambda_3 = 3$: $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

orthogonal (not an accident; Thm 1, coming up) normalize: $\|\vec{v}_1\| = \sqrt{2}$, $\|\vec{v}_2\| = \sqrt{6}$, $\|\vec{v}\|_3 = \sqrt{3}$

$$\vec{u}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\boxed{\begin{gathered} P = \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{\text{orthonormal cols}}, \quad D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ \therefore P \text{ is an orthogonal matrix, } P^{-1} = P^T \\ A = PDP^{-1} \quad \text{or} \quad A = PDP^T \end{gathered}}$$

<u>Thm 1</u> Let A be symmetric. Let λ_1, λ_2 be eigvals of A with $\lambda_1 \neq \lambda_2$. Let $\vec{v}_1, \vec{v}_2$ be eigvects. corr. to λ_1, λ_2 respectively. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

<u>Pf</u>:

A is symmetric, so $A^T = A$

$$\begin{aligned} \lambda_1 \vec{v}_1 \cdot \vec{v}_2 &= \lambda_1 \vec{v}_1^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 \\ &= (A\vec{v}_1)^T \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = \vec{v}_1^T A \vec{v}_2 \\ &= \vec{v}_1^T(\lambda_2 \vec{v}_2) = \lambda_2 \vec{v}_1 \cdot \vec{v}_2 \end{aligned}$$

$$\therefore \underbrace{(\lambda_1 - \lambda_2)}_{\text{not zero}} \vec{v}_1 \cdot \vec{v}_2 = 0 \implies \vec{v}_1 \cdot \vec{v}_2 = 0. \quad \square$$

A is <u>orthogonally diagonalizable</u> if there is an orthogonal P and diagonal D such that $A = PDP^{-1} = PDP^T$.

<u>Thm 2</u> Let A be $n \times n$. Then A is orthogonally diagonalizable iff A is symmetric.

<u>Expl 3</u>

Orthogonally diagonalize $A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

$$\det(A - \lambda I) = \cdots = -\lambda^3 + 12\lambda^2 - 21\lambda - 98$$
$$= -(\lambda - 7)^2(\lambda + 2)$$

$\lambda_1 = 7$: (eig. space will have dim. 2 by Thm 2)

$$\underbrace{\cdots \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}}_{\text{lin. indep, but not orthogonal}} \leftarrow \text{basis for eigspace}$$

need orthogonal... use Gram-Schmidt

$\vec{z}_1 = \vec{v}_1$

$$\vec{z}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} - \frac{-1/2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \\ 1/4 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 1 \\ 1/4 \end{pmatrix}$$

$\lambda = -2$:

$$\ldots \vec{v}_3 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}$$ orthogonal to others by Thm 1

orthogonal eigvector basis:

$$\vec{z}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{z}_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

orthonormal eigvector basis:

$$\|\vec{z}_1\| = \sqrt{2}, \quad \|\vec{z}_2\| = \sqrt{18}, \quad \|\vec{v}_3\| = \sqrt{9} = 3$$

$$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad A = PDP^{-1} = PDP^T}$$

"spectrum" is the set of eigenvalues

<u>Thm 3</u> (The spectral theorem for symmetric matrices)

Let A be $n \times n$ and symmetric. Then:

(a) A has n real eigvals, counting multiplicities.

(b) For each eigval λ,

$$\dim(\text{eig. space}) = \text{multiplicity of } \lambda.$$

(c) Eigenspaces are orthogonal.

(d) A is orthogonally diagonalizable.

<u>Spectral decomposition</u>

$$A = PDP^T, \quad P = (\vec{u}_1 \cdots \vec{u}_n), \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$PD = P(\lambda_1 \vec{e}_1 \cdots \lambda_n \vec{e}_n) = (\lambda_1 P\vec{e}_1 \cdots \lambda_n P\vec{e}_n)$$

$$= (\lambda_1 \vec{u}_1 \cdots \lambda_n \vec{u}_n)$$

$$PDP^T = (\lambda_1 \vec{u}_1 \cdots \lambda_n \vec{u}_n) \begin{pmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{pmatrix}$$

$$A = \underbrace{\lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T}_{\text{spectral decomposition}}$$

$$\underbrace{\vec{u}_k \vec{u}_k^T}_{\text{a projection matrix; projects onto } \vec{u}_k} \vec{x} = \vec{u}_k \big(\underset{\uparrow \text{ unit vector}}{\vec{u}_k} \cdot \vec{x} \big) = \text{Proj}_{\vec{u}_k} \vec{x}$$

Expl 3 again

$$A = \begin{array}{c} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{pmatrix} \\ \begin{matrix} \uparrow & \quad\uparrow & \quad\uparrow \\ \vec{u}_1 & \quad\vec{u}_2 & \quad\vec{u}_3 \end{matrix} \end{array} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} P^T$$

$$\vec{u}_1 \vec{u}_1^T = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \underbrace{\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}}_{\text{projects onto } \vec{u}_1}$$

$$\vec{u}_2 \vec{u}_2^T = \begin{pmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{pmatrix} = \begin{pmatrix} 1/18 & -2/9 & -1/18 \\ -2/9 & 8/9 & 2/9 \\ -1/18 & 2/9 & 1/18 \end{pmatrix}$$

$$\vec{u}_3 \vec{u}_3^T = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{pmatrix}$$

$$\boxed{A = 7\vec{u}_1\vec{u}_1^T + 7\vec{u}_2\vec{u}_2^T - 2\vec{u}_3\vec{u}_3^T}$$

Supplemental:

$$A = 7\,(\underbrace{\vec{u}_1\vec{u}_1^T + \vec{u}_2\vec{u}_2^T}_{\text{projects onto eigspace for } \lambda = 7}) - 2\,\vec{u}_3\vec{u}_3^T$$

($\{\vec{u}_1, \vec{u}_2\}$ an orthonormal basis)

In general...

$A = n \times n$ symmetric

$\lambda_1, \dots, \lambda_p$ distinct eigvals.

$T_k : \mathbb{R}^n \to \mathbb{R}^n$ orthog. proj. onto eigsp. corr. to λ_k

P_k : matrix for T_k

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_p P_p$$