6.4 The Gram-Schmidt process

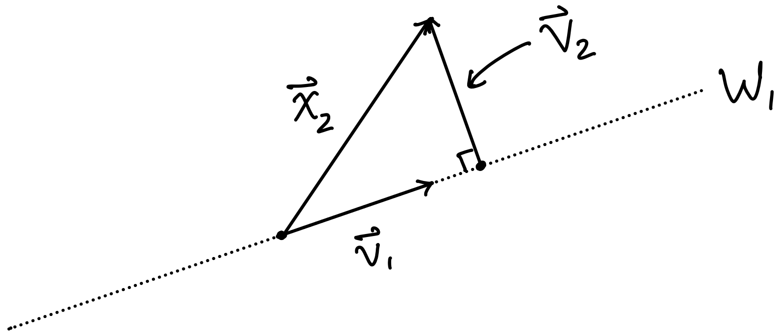
Expl 2

$$\vec{x} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$W = \text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. Find an orthogonal basis for W.

$$\vec{v}_1 = \vec{x}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$W_1 = \text{Span}\{\vec{v}_1\} = \text{Span}\{\vec{x}_1\}$$



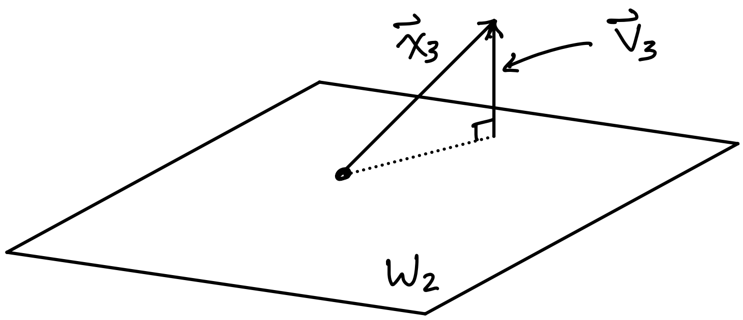
$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1}\vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= \begin{pmatrix}0\\1\\1\\1\end{pmatrix} - \underset{\frac{3}{4}}{\cancel{\frac{0+1+1+1}{1+1+1+1}}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-3/4\\1/4\\1/4\\1/4\end{pmatrix}$$

Can use any scalar multiple of this for $\vec{v}_2$

$$\boxed{\vec{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

$W_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{x}_1, \vec{x}_2\}$



$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2}\vec{x}_3$

$$\text{proj}_{W_2}\vec{x}_3 = \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \overset{\frac{1}{2}}{\cancel{\frac{2}{4}}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \overset{\frac{1}{6}}{\cancel{\frac{2}{12}}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \text{ take } \vec{v}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

Orthogonal basis for W:

$$\boxed{\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}\right\}}$$

Thm 11 (The Gram-Schmidt process)

Let $\{\vec{x}_1, \dots, \vec{x}_p\}$ be a basis for $W \subset \mathbb{R}^n$. Define

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}$$

Then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal basis for W and, for all k, $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$.

To construct an orthonormal basis, first construct an orthogonal basis, then normalize the vectors.

QR factorization

(used in many algorithms, e.g. see Exer. 31-32 in §5.2)

$$\overset{m \times n}{A} = (\underbrace{\vec{a}_1 \cdots \vec{a}_n}_{\text{suppose these are lin. indep.}})$$

$\{\vec{a}_1, \dots, \vec{a}_n\}$ is a basis for $\text{Col}\, A \subset \mathbb{R}^m$

Use Gram-Schmidt to make

$\{\vec{u}_1, \ldots, \vec{u}_n\}$, an ON basis for Col A

$\underset{m\times n}{Q} := (\vec{u}_1 \cdots \vec{u}_n)$

$\underset{n\times n}{R} := Q^T A$, R will be upper triangular (Thm 12)

Q has ON cols, so $Q^T Q = \underset{n\times n}{I}$

Thm 12 $\Rightarrow$ $\underset{\substack{m\times n\\ \text{lin. indep. cols.}}}{A} = \underset{\substack{m\times n\\ \text{ON cols}}}{Q} \; \underset{\substack{n\times n\\ \text{upper triangular}}}{R}$

<u>Expl 4</u> Find a QR factorization for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

lin. indep. cols, basis for Col A

Make an ON basis

Gram-Schmidt gives orthogonal basis:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{(Expl 2)}$$

lengths: $\sqrt{4} = 2$ $\quad \sqrt{12}$ $\quad \sqrt{6}$

normalize and put into Q:

$$Q = \begin{pmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{pmatrix}$$

upper triangular

Find R:

$$Q^T A = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{pmatrix}$$

$A = QR$

$$= \boxed{\begin{pmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{pmatrix}}$$