6.3 Orthogonal projections

Thm 8 (Orthogonal decomposition thm)

Let $W \subset \mathbb{R}^n$ be a subspace. Each $\vec{y} \in \mathbb{R}^n$ can be uniquely written as $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y} \in W$ and $\vec{z} \in W^\perp$.

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal basis for W, then

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \vec{u}_p$$

and $\vec{z} = \vec{y} - \hat{y}$.

the "orthogonal projection of $\vec{y}$ onto W"

$\hat{y}$ is also written as "$\text{proj}_W \vec{y}$"

Expl

$\vec{u}_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ are orthogonal.

$W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$. $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Write $\vec{y}$ as a sum of a vector in W and a vector in $W^\perp$.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

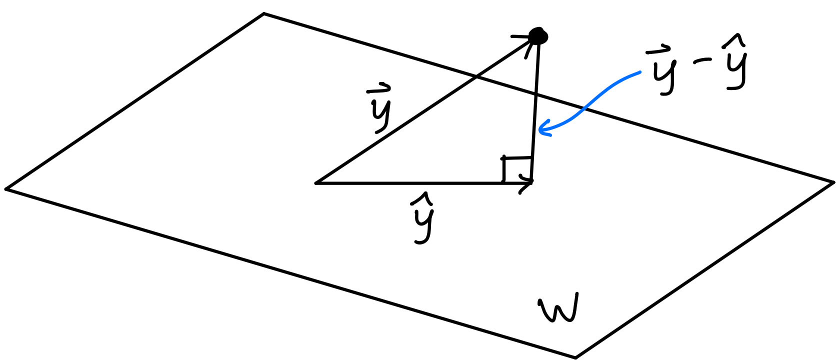
$$= \frac{2+10-3}{4+25+1} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \frac{-2+2+3}{4+1+1} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{9}{30}\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \frac{15}{30}\cancel{\frac{3}{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix} 18-30 \\ 45+15 \\ -9+15 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} -12 \\ 60 \\ 6 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 0 \\ 14/5 \end{pmatrix}$$

$$\boxed{\vec{y} = \underbrace{\begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix}}_{\text{in } W} + \underbrace{\begin{pmatrix} 7/5 \\ 0 \\ 14/5 \end{pmatrix}}_{\text{in } W^{\perp}}}$$



$\|\vec{y} - \hat{y}\|$ is the distance from $\vec{y}$ to W

Thm 9

If $\vec{v} \in W$ and $\vec{v} \neq \hat{y}$, then $\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$.

($\hat{y}$ is the closest point in W to $\vec{y}$)

Pf:

$$\|\vec{y}-\vec{v}\|^2 = \|(\vec{y}-\hat{y})+(\hat{y}-\vec{v})\|^2$$

$$= \|\vec{y}-\hat{y}\|^2 + 2\underbrace{(\vec{y}-\hat{y})}_{\text{in } W^\perp}\cdot\underbrace{(\hat{y}-\vec{v})}_{\text{in } W} + \|\hat{y}-\vec{v}\|^2$$

$$= \|\vec{y}-\hat{y}\|^2 + 0 + \underbrace{\|\hat{y}-\vec{v}\|^2}_{\vec{v}\neq\hat{y},\ \therefore\ \hat{y}-\vec{v}\neq\vec{0}\quad \therefore \|\hat{y}-\vec{v}\|>0}$$

$$> \|\vec{y}-\hat{y}\|^2. \quad \square$$

Expl 4

Find the distance from $\vec{y}=\begin{pmatrix}-1\\-5\\10\end{pmatrix}$ to $W=\text{Span}\{\vec{u}_1,\vec{u}_2\}$, where $\vec{u}_1=\begin{pmatrix}5\\-2\\1\end{pmatrix}$, $\vec{u}_2=\begin{pmatrix}1\\2\\-1\end{pmatrix}$.

$\vec{u}_1\cdot\vec{u}_2 = 5-4-1=0$,

so $\{\vec{u}_1,\vec{u}_2\}$ is an orthogonal basis for W

$$\therefore\quad \hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

$$= \frac{-5+10+10}{25+4+1}\begin{pmatrix}5\\-2\\1\end{pmatrix} + \frac{-1-10-10}{1+4+1}\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$= \overset{\frac{1}{2}}{\cancel{\frac{15}{30}}}\begin{pmatrix}5\\-2\\1\end{pmatrix} - \overset{\frac{7}{2}}{\cancel{\frac{21}{6}}}\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{2} - \frac{7}{2} \\ -1 - 7 \\ \frac{1}{2} + \frac{7}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \\ 4 \end{pmatrix}$$

$$\|\vec{y} - \hat{y}\|^2 = \left\| \begin{pmatrix} -1 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} -1 \\ -8 \\ 4 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \right\|^2$$

$$= 0 + 9 + 36 = 45 \quad , \quad \|\vec{y} - \hat{y}\| = \sqrt{45} = \boxed{3\sqrt{5}}$$

Thm 10

Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthonormal basis for a subspace $W \subset \mathbb{R}^n$. Let $U = (\vec{u}_1 \cdots \vec{u}_p)$. Then $\text{proj}_W \vec{y} = UU^T \vec{y}$.

Pf:

$$\text{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\underset{1}{\cancel{\vec{u}_1 \cdot \vec{u}_1}}} \vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\underset{1}{\cancel{\vec{u}_p \cdot \vec{u}_p}}} \vec{u}_p$$

$$= (\vec{u}_1^T \vec{y}) \vec{u}_1 + \cdots + (\vec{u}_p^T \vec{y}) \vec{u}_p$$

$$= U \begin{pmatrix} \vec{u}_1^T \vec{y} \\ \vdots \\ \vec{u}_p^T \vec{y} \end{pmatrix} = UU^T \vec{y}. \quad \square$$