6.2 Orthogonal sets

$\{\vec{u}_1, \ldots, \vec{u}_p\} \subset \mathbb{R}^n$ is an <u>orthogonal set</u> if

$\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$.

<u>Expl 1</u>

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -1/2 \\ -2 \\ 7/2 \end{pmatrix}$$

Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

$\vec{u}_1 \cdot \vec{u}_2 = -3 + 2 + 1 = 0$ ✓

$\vec{u}_1 \cdot \vec{u}_3 = -\frac{3}{2} - 2 + \frac{7}{2} = 0$ ✓

$\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{2} - 4 + \frac{7}{2} = 0$ ✓

<u>Thm 4</u> If $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors, then S is lin. indep; and so S is a basis for Span S.

<u>Pf</u>: Suppose $c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = \vec{0}$. Fix $i \in \{1, \ldots, p\}$.

Then

$$0 = \vec{u}_i \cdot \vec{0} = \vec{u}_i \cdot (c_1\vec{u}_1 + \cdots + c_p\vec{u}_p)$$

$$= c_1 \vec{u}_i \cdot \vec{u}_1 + \cdots + c_p \vec{u}_i \cdot \vec{u}_p .$$

Since S is an orthogonal set, $\vec{u}_i \cdot \vec{u}_j = 0$ when $j \neq i$ and $\vec{u}_i \cdot \vec{u}_i = \|\vec{u}_i\|^2$. Thus,

$$0 = c_i \|\vec{u}_i\|^2 .$$

Since $\vec{u}_i \neq \vec{0}$, we have $\|\vec{u}_i\|^2 > 0$. Therefore, $c_i = 0$. This is true for every i, so S is lin. indep. $\square$

$W \subset \mathbb{R}^n$ a subspace

An <u>orthogonal basis</u> for W is a basis for W that is also an orthogonal set.

<u>Thm 5</u> Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace $W \subset \mathbb{R}^n$. If $\vec{y} \in W$, then

$$\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p ,$$

where $c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$.

can find weights without row reduction

Pf: $\vec{y}\cdot\vec{u}_j = (c_1\vec{u}_1 + \cdots + c_p\vec{u}_p)\cdot\vec{u}_j$

$$= c_1\vec{u}_1\cdot\vec{u}_j + \cdots + c_p\vec{u}_p\cdot\vec{u}_j$$

$$= c_j\,\vec{u}_j\cdot\vec{u}_j .$$

Solve: $c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$. $\square$

Expl 2

$$\vec{u}_1 = \begin{pmatrix}3\\1\\1\end{pmatrix},\quad \vec{u}_2 = \begin{pmatrix}-1\\2\\1\end{pmatrix},\quad \vec{u}_3 = \begin{pmatrix}-1/2\\-2\\7/2\end{pmatrix}$$

from Expl 1

$\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for $\mathbb{R}^3$.

$\vec{y} = \begin{pmatrix}6\\1\\-8\end{pmatrix}$. Find $(\vec{y})_{\mathcal{B}}$.

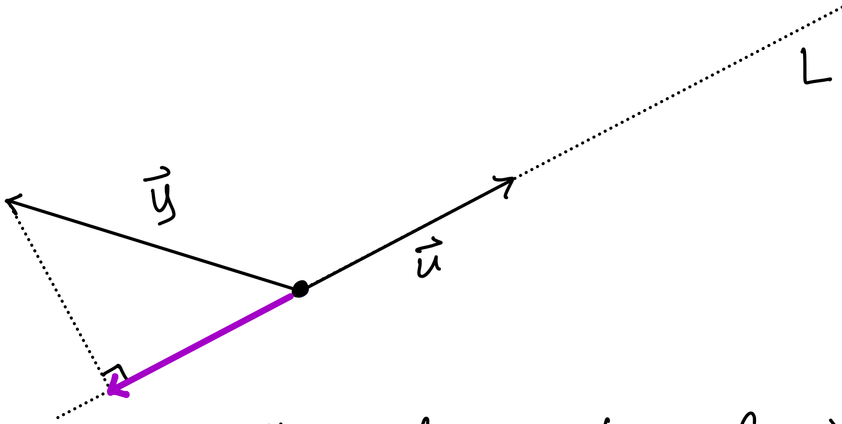
$$\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$$

$$c_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{18+1-8}{9+1+1} = \frac{11}{11} = 1 \qquad \vec{u}_3 = \begin{pmatrix}-1/2\\-2\\7/2\end{pmatrix}$$

$$c_2 = \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{-6+2-8}{1+4+1} = \frac{-12}{6} = -2$$

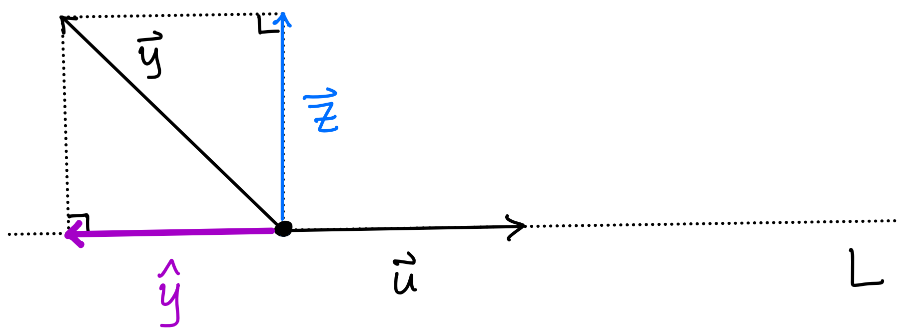
$$c_3 = \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{-3-2-28}{\frac{1}{4}+4+\frac{49}{4}} = \frac{-33}{\frac{66}{4}} = -\cancel{33}\cdot\frac{4}{\cancel{66}_{\,2}} = -2$$

$$\boxed{(\vec{y})_{\mathcal{B}} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}}$$



orthogonal projection of $\vec{y}$ onto $\vec{u}$
or onto L

notation: $\boxed{\hat{y}}$ or $\boxed{\text{proj}_L \vec{y}}$



$\hat{y} = \alpha \vec{u}$, $\alpha = ?$

$\hat{y} + \vec{z} = \vec{y}$

$\vec{z} \perp \vec{u}$, so...

$$\vec{z} \cdot \vec{u} = 0$$

$$(\vec{y} - \hat{y}) \cdot \vec{u} = 0$$

$$(\vec{y} - \alpha \vec{u}) \cdot \vec{u} = 0$$

$$\vec{y} \cdot \vec{u} - \alpha\, \vec{u} \cdot \vec{u} = 0$$

$$\alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

$$\boxed{\hat{y} = \operatorname{proj}_L \vec{y} = \underbrace{\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}}_{\text{a scalar}} \underbrace{\vec{u}}_{\text{a vector}}}$$

If $W = \operatorname{Span}\{\vec{u}\}$, then

$\hat{y} \in W$, $\vec{z} \in W^{\perp}$, and $\vec{y} = \hat{y} + \vec{z}$

<u>Expl 3</u>

$\vec{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Write $\vec{y}$ as a sum of two vectors, one in $\operatorname{Span}\{\vec{u}\}$ and one orthogonal to $\vec{u}$.

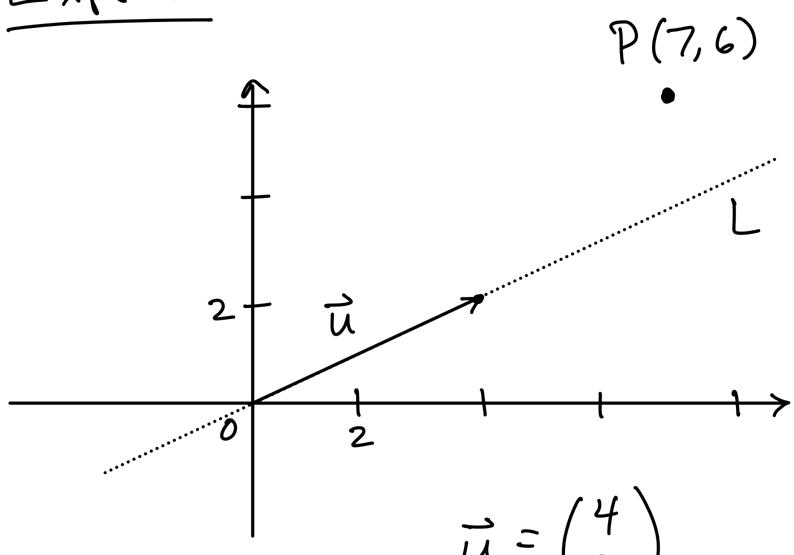
$$\vec{y} = \underset{\uparrow \in \text{Span}\{\vec{u}\}}{\hat{y}} + \vec{z} \leftarrow \text{orthogonal to } \vec{u}$$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{28+12}{16+4}\begin{pmatrix}4\\2\end{pmatrix} = \frac{\overset{2}{\cancel{40}}}{\cancel{20}}\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{pmatrix}7\\6\end{pmatrix} - \begin{pmatrix}8\\4\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

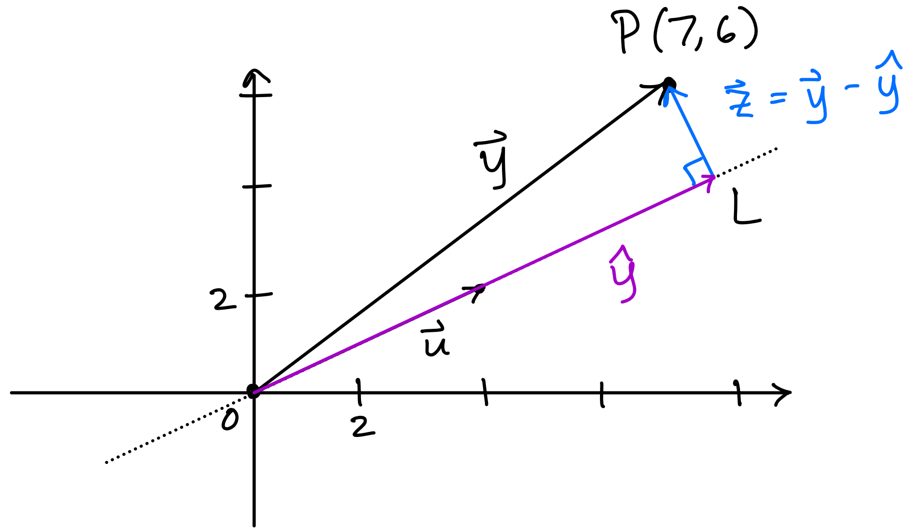
$$\boxed{\vec{y} = \underset{\uparrow\ \text{in Span}\{\vec{u}\}}{\begin{pmatrix}8\\4\end{pmatrix}} + \underset{\uparrow\ \text{orthogonal to } \vec{u}}{\begin{pmatrix}-1\\2\end{pmatrix}}}$$

Expl 4



Find the distance from P to L

$$\vec{u} = \begin{pmatrix}4\\2\end{pmatrix}$$



$$\vec{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

$$\text{distance} = \|\vec{y} - \hat{y}\| = \left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\| = \sqrt{1+4} = \boxed{\sqrt{5}}$$

An <u>orthonormal set/basis</u> is an orthogonal set/basis of unit vectors.

<u>Expl 5</u>

$$\vec{v}_1 = \begin{pmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{pmatrix}$$

Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \frac{1}{\sqrt{66}}(-3+2+1) = 0 \checkmark$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{66}}\begin{pmatrix}-1\\-4\\7\end{pmatrix} = \frac{1}{\sqrt{11}\sqrt{66}}(-3-4+7) = 0 \checkmark$$

$$\vec{v}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix} \cdot \frac{1}{\sqrt{66}}\begin{pmatrix}-1\\-4\\7\end{pmatrix} = \frac{1}{\sqrt{6}\sqrt{66}}(1-8+7) = 0 \checkmark$$

So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set

$\Rightarrow$ it is lin. indep

$\Rightarrow$ it's a basis for $\mathbb{R}^3$ (since $\dim \mathbb{R}^3 = 3$)

Is it orthonormal?

$$\|\vec{v}_1\|^2 = \vec{v}_1 \cdot \vec{v}_1 = \frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix} = \frac{1}{11}(9+1+1) = \frac{11}{11} = 1 \checkmark$$

$$\vec{v}_2 \cdot \vec{v}_2 = \frac{1}{6}(1+4+1) = \frac{6}{6} = 1 \checkmark$$

$$\vec{v}_3 \cdot \vec{v}_3 = \frac{1}{66}(1+16+49) = \frac{66}{66} = 1 \checkmark$$

YES

Thm 6

Let U be an $m \times n$ matrix. Then

U has orthonormal columns iff $U^T U = I$.

$n \times n$

Pf: $U = (\vec{u}_1 \cdots \vec{u}_n)$

$$U^T U = \begin{pmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{pmatrix} (\vec{u}_1 \cdots \vec{u}_n)$$

$$= \underbrace{\begin{pmatrix} \vec{u}_1^T \vec{u}_1 & \vec{u}_1^T \vec{u}_2 & \cdots & \vec{u}_1^T \vec{u}_n \\ \vec{u}_2^T \vec{u}_1 & \vec{u}_2^T \vec{u}_2 & \cdots & \vec{u}_2^T \vec{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_n^T \vec{u}_1 & \vec{u}_n^T \vec{u}_2 & \cdots & \vec{u}_n^T \vec{u}_n \end{pmatrix}}$$

This is I iff $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthonormal set. $\square$

Thm 7 If U is $m \times n$ w/ orthormal cols, then

(a) $\|U\vec{x}\| = \|\vec{x}\|$

(b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

(c) $(U\vec{x}) \cdot (U\vec{y}) = 0$ iff $\vec{x} \cdot \vec{y} = 0$

$\vec{x} \mapsto U\vec{x}$ preserves length and orthogonality

Pf:

(b): $(U\vec{x}) \cdot (U\vec{y}) = (U\vec{x})^T (U\vec{y}) = \vec{x}^T U^T U \vec{y}$

$= \vec{x}^T I \vec{y} = \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}$ ✓

(b) $\Rightarrow$ (c) ✓

(a): $\|U\vec{x}\|^2 = (U\vec{x})\cdot(U\vec{x}) = \vec{x}\cdot\vec{x} = \|\vec{x}\|^2$. $\square$

Expl 6

$$U = \begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix}$$ has orthonormal cols.

$\vec{x} = \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix}$. Verify that $U^TU = I$ and $\|U\vec{x}\| = \|\vec{x}\|$.

$$U^TU = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} + 0 & \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 \\ \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 & \frac{4}{9} + \frac{4}{9} + \frac{1}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ✓$$

$$\|\vec{x}\|^2 = \left\|\begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix}\right\|^2 = 2 + 9 = 11$$

$$\|\vec{x}\| = \sqrt{11}$$

$$U\vec{x} = \begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 1-2 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\|U\vec{x}\|^2 = 9+1+1 = 11$$

$$\|U\vec{x}\| = \sqrt{11} \checkmark$$

confusing clash of terminology

An <u>orthogonal matrix</u> is a square matrix U with orthonormal columns.

- U is orthogonal iff $U^{-1} = U^T$
- if U is orthogonal, then it also has orthonormal rows

Exercises 35, 36

<u>Expl 7</u>

$$U = (\underbrace{\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3}_{\text{from Expl 5}}) = \underbrace{\begin{pmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{pmatrix}}_{\text{orthonormal cols}}$$

is an orthogonal matrix