6.1 Inner product, length, and orthogonality

$\vec{u}, \vec{v} \in \mathbb{R}^n$

<u>inner product</u> or <u>dot product</u> :

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = (u_1 \cdots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

This is a 1x1 matrix, but we drop the parentheses and think of it as a scalar.

$$= (u_1 v_1 + u_2 v_2 + \cdots + u_n v_n)$$

Expl

$\vec{u} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$. Find $\vec{u} \cdot \vec{v}$ and $\vec{v} \cdot \vec{u}$.

$\vec{u} \cdot \vec{v} = 2(3) + (-5)(2) + (-1)(-3) = 6 - 10 + 3 = -1$

$\vec{v} \cdot \vec{u} = 3(2) + 2(-5) + (-3)(-1) = 6 - 10 + 3 = -1$

Thm 1

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(c) $(cu) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

(d) $\underbrace{\vec{u} \cdot \vec{u}}_{u_1^2 + u_2^2 + \cdots + u_n^2} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0}$

The <u>length</u> or <u>norm</u> of $\vec{u}$ is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$$

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Agrees w/ geometric length when $n=2$ or $n=3$.

$n=1$: $\quad \|u\| = \sqrt{u^2} = |u|$ many people get this wrong

Length is absolute value when $n=1$.

Formulas:

- $\|c\vec{u}\| = |c|\,\|\vec{u}\|$
- $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$

  Derivation:

  $$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
  $$= \underbrace{\vec{u}\cdot\vec{u}}_{\|\vec{u}\|^2} + 2\,\vec{u}\cdot\vec{v} + \underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2}$$

- $\left\|\sum_{i=1}^{p}\vec{u}_i\right\|^2 = \sum_{i=1}^{p}\|\vec{u}_i\|^2 + 2\sum_{i<j}\vec{u}_i\cdot\vec{u}_j$

  Derivation:

$$\left\| \sum_{i=1}^{p} \vec{u}_i \right\|^2 = \left(\sum_{i=1}^{p} \vec{u}_i \right) \cdot \left(\sum_{j=1}^{p} \vec{u}_j \right)$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} \vec{u}_i \cdot \vec{u}_j$$

$$= \sum_{i=1}^{p} \underbrace{\vec{u}_i \cdot \vec{u}_i}_{\|\vec{u}_i\|^2} + \underbrace{\sum_{i \neq j} \vec{u}_i \cdot \vec{u}_j}_{\text{only sum over } i<j \text{ and multiply by } 2}$$

$\vec{u}$ is a <u>unit vector</u> if $\|\vec{u}\| = 1$

If $\vec{v} \neq \vec{0}$, we can <u>normalize</u> $\vec{v}$ by creating a new vector, $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$. Then $\vec{u}$ is a unit vector with the $\underbrace{\text{"same direction"}}$ as $\vec{v}$.

this makes sense when $n \leq 3$;
we use the same language for $n > 3$

<u>Expl 2</u>

$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix}$. Find a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

$$\|\vec{v}\|^2 = 1^2 + (-2)^2 + 2^2 + 0^2 = 1 + 4 + 4 = 9$$

$$\|\vec{v}\| = 3$$

$$\vec{u} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{pmatrix}}$$

Expl 3

$\vec{x} = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$, $W = \text{Span}\{\vec{x}\}$. Find a basis $\{\vec{z}\}$ for W, where $\vec{z}$ is a unit vector.

Need $\vec{z} = c\vec{x}$ where $c \neq 0$, and $\|\vec{z}\| = 1$

choose this

Can take $c = \frac{1}{\|\vec{x}\|}$ or $c = -\frac{1}{\|\vec{x}\|}$

$$\|\vec{x}\|^2 = \frac{4}{9} + 1 = \frac{13}{9}, \quad \|\vec{x}\| = \frac{\sqrt{13}}{3}$$

$$\vec{z} = \frac{3}{\sqrt{13}}\begin{pmatrix} 2/3 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{pmatrix}}$$

other answer:

$$\begin{pmatrix} -2/\sqrt{13} \\ -3/\sqrt{13} \end{pmatrix}$$

The distance between $\vec{u}$ and $\vec{v}$ is

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

Agrees w/ geometric distance when $n \leq 3$.

When $n=1$, this is distance on the number line.

Expl 4

$\vec{u} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the distance between $\vec{u}$ and $\vec{v}$.

$$\|\vec{u} - \vec{v}\|^2 = \left\|\begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\|^2 = \left\|\begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\|^2$$

$$= 16 + 1 = 17$$

$$\|\vec{u} - \vec{v}\| = \boxed{\sqrt{17}}$$

$\vec{u}$ and $\vec{v}$ are orthogonal or perpendicular

if $\vec{u} \cdot \vec{v} = 0$.

Agrees w/ geometry when $n \leq 3$.

Thm 2 (Pythagorean Thm)

$\vec{u}$ and $\vec{v}$ are orthogonal iff $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

Pf: $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$. $\therefore$

$\vec{u}, \vec{v}$ orthogonal iff $\vec{u}\cdot\vec{v} = 0$
iff $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$. $\square$

$W \subset \mathbb{R}^n$ a subspace

$\vec{z}$ is orthogonal to W if $\vec{z}\cdot\vec{w} = 0 \quad \forall \vec{w} \in W$

$W^\perp = \{\vec{z} \in \mathbb{R}^n : \vec{z} \text{ is orthogonal to } W\}$

orthogonal complement of W ("W perp")

- If $W = \operatorname{Span} S$, then $W^\perp = \{\vec{z} : \vec{z}\cdot\vec{x} = 0 \ \forall \vec{x} \in S\}$
- $W^\perp$ is a subspace
- $\dim W + \dim W^\perp = n$

covered in the exercises

<u>Thm 3</u> Let A be an $m \times n$ matrix. Then $\mathrm{Nul}\,A = (\mathrm{Row}\,A)^\perp$ and $\mathrm{Nul}(A^T) = (\mathrm{Col}\,A)^\perp$.

<u>Pf:</u> Let $\vec{u} \in \mathrm{Nul}\,A$. Let $\vec{v} \in \mathrm{Row}\,A$. Write $A = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{pmatrix}$. Then $\vec{v} = c_1\vec{r}_1 + \cdots + c_m\vec{r}_m$ for some $c_1, \ldots, c_m$. Since $\vec{0} = A\vec{u} = \begin{pmatrix} \vec{r}_1^{\,T}\vec{u} \\ \vdots \\ \vec{r}_m^{\,T}u \end{pmatrix}$, we have

$\vec{r}_i \cdot \vec{u} = \vec{r}_i^{\,T}\vec{u} = 0$ for all i. Thus,

$$\begin{aligned} \vec{v}\cdot\vec{u} &= (c_1\vec{r}_1 + \cdots + c_m\vec{r}_m)\cdot\vec{u} \\ &= c_1\vec{r}_1\cdot\vec{u} + \cdots + c_m\vec{r}_m\cdot\vec{u} = 0. \end{aligned}$$

In other words, $\vec{u}$ and $\vec{v}$ are orthogonal. Since $\vec{v} \in \mathrm{Row}\,A$ was arbitrary, this shows that $\vec{u} \in (\mathrm{Row}\,A)^\perp$. Since this is true for every $\vec{u} \in \mathrm{Nul}\,A$, we have $\mathrm{Nul}\,A \subset (\mathrm{Row}\,A)^\perp$.

Now let $\vec{u} \in (\mathrm{Row}\,A)^\perp$. Since $\vec{r}_i \in \mathrm{Row}\,A$ for all i, we have $\vec{r}_i \cdot \vec{u} = 0$ for all i. Thus,

$$A\vec{u} = \begin{pmatrix} \vec{r}_1^{\,T}\vec{u} \\ \vdots \\ \vec{r}_m^{\,T}\vec{u} \end{pmatrix} = \begin{pmatrix} \vec{r}_1\cdot\vec{u} \\ \vdots \\ \vec{r}_m\cdot\vec{u} \end{pmatrix} = \vec{0},$$

so that $\vec{u} \in \operatorname{Nul} A$. Since $\vec{u}$ was arbitrary, $(\operatorname{Row} A)^\perp \subset \operatorname{Nul} A$. Combined with $\operatorname{Nul} A \subset (\operatorname{Row} A)^\perp$, which we proved earlier, this gives
$\operatorname{Nul} A = (\operatorname{Row} A)^\perp$.

Finally, applying the above to A^T, we have
$\operatorname{Nul}(A^T) = (\operatorname{Row}(A^T))^\perp = (\operatorname{Col} A)^\perp$. $\square$