5.3 Diagonalization

Expl 1

$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$. Find a formula for D^k.

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ 0 & 27 \end{pmatrix} \quad \cdots$$

$$\boxed{D^k = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}}$$

In general, if D is diagonal, then

$$D^k = \begin{pmatrix} d_{11}^k & & 0 \\ & d_{22}^k & \\ 0 & & \ddots & \\ & & & d_{nn}^k \end{pmatrix}$$

Expl 2

$$A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$$

Given: $A = PDP^{-1}$. Find a formula for A^k.

$$A^2 = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PDDP^{-1} = PD^2P^{-1}$$

$$A^3 = PD^2\cancel{P^{-1}P}DP^{-1} = PD^3P^{-1}$$

$$\vdots$$

$$A^k = PD^kP^{-1}$$

$$\det P = -2-(-1) = -1$$

$$P^{-1} = \frac{1}{-1}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2(5^k) & 5^k \\ -3^k & -3^k \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2(5^k)-3^k & 5^k-3^k \\ -2(5^k)+2(3^k) & -5^k+2(3^k) \end{pmatrix}}$$

If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ and each $\vec{v}_k$ is an eig. vect. of A, then $\mathcal{B}$ is an eigenvector basis for $\mathbb{R}^n$.

A is diagonalizable if there is an invertible P and a diagonal D s.t. $A = PDP^{-1}$.
(P and D are not unique.)

P to the left of D.
important

Thm 5(a)

Suppose A is diagonalizable. Write $A = \underbrace{PD}P^{-1}$,
$P = (\vec{v}_1 \cdots \vec{v}_n)$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. Then each

λ_k is an eig. val. of A, each $\vec{v}_k$ is an eig. vect. of A corresponding to λ_k, and $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an eig. vect. basis of $\mathbb{R}^n$.

Pf: $\quad P = (\vec{v}_1 \cdots \vec{v}_n)$

$$AP = A(\vec{v}_1 \cdots \vec{v}_n) = (A\vec{v}_1 \cdots A\vec{v}_n)$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = (\lambda_1\vec{e}_1 \cdots \lambda_n\vec{e}_n)$$

$P\vec{e}_k = \vec{v}_k$ for all k

$$PD = (P(\lambda_1 \vec{e}_1) \cdots P(\lambda_n \vec{e}_n))$$

$$= (\lambda_1 \vec{v}_1 \cdots \lambda_n \vec{v}_n)$$

$$AP = PDP^{-1}P = PD$$

$\therefore$ $(A\vec{v}_1 \cdots A\vec{v}_n) = (\lambda_1 \vec{v}_1 \cdots \lambda_n \vec{v}_n)$

$\therefore$ $A\vec{v}_k = \lambda_k \vec{v}_k$ for all k.

Each λ_k is an eig. val. ✓

" $\vec{v}_k$ is an eig. vect. corr. to λ_k ✓

P is invertible, so cols $\vec{v}_1, \ldots, \vec{v}_n$ are lin. indep.

$\therefore$ $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. $\square$

Thm 5(b)

Let A be $n \times n$. Suppose there is an eig. vect. basis for $\mathbb{R}^n$. Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be such an eig. vect. basis. Let λ_k be the eig. val. corr. to $\vec{v}_k$. Define $P = (\vec{v}_1 \cdots \vec{v}_n)$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Then $A = PDP^{-1}$.

Pf:

$$AP = (A\vec{v}_1 \cdots A\vec{v}_n) = (\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n)$$
$$= (\lambda_1 P\vec{e}_1 \cdots \lambda_n P\vec{e}_n)$$
$$= P(\lambda_1\vec{e}_1 \cdots \lambda_n\vec{e}_n)$$
$$= PD.$$

Cols of P are lin. indep. $\therefore$ P invertible.

$\therefore$ $A = APP^{-1} = PDP^{-1}$. $\square$

SUMMARY A is diagonalizable iff there are n lin. indep. eigenvectors.

Expl 3

Diagonalize $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ or explain why it's not diagonalizable.

Find eigvals:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\big((-5-\lambda)(1-\lambda)+9\big) - 3\big(-3(1-\lambda)+9\big)$$
$$+ 3\big(-9-(-5-\lambda)(3)\big)$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) \cancel{-3(3\lambda+6)} \cancel{+3(3\lambda+6)}$$

$$= -(\lambda-1)(\lambda+2)^2 = 0$$

eigvals: $\lambda_1 = 1$ (mult. 1)

$\lambda_2 = -2$ (mult. 2)

Find eigvects:

$\lambda_1 = 1$:

$(A - 1I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \leftrightarrow R_3 \\ R_2 + R_1 \to R_2 \\ \frac{1}{3}R_1 \to R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right)$$

$$\begin{array}{l} R_3 + R_2 \to R_3 \\ -\frac{1}{3}R_2 \to R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 - R_2 \to R_1 \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

eig. space for $\lambda_1 = 1$ is $\left\{ t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$

$= \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

one eigvect : $\underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\vec{v}_1}$. Need two more

$\lambda_2 = -2$:

$(A - (-2)I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

eig. space for $\lambda_2 = -2$ is $\text{Span} \left\{ \underbrace{\overset{\vec{v}_2}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}, \overset{\vec{v}_3}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}}_{\text{two more eigvects}} \right\}$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are lin. indep (check)

$$P = (\vec{v}_1\ \vec{v}_2\ \vec{v}_3) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

eigvals must match the eigvects

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$A = PDP^{-1}$$ (can check)

Expl 4

Diagonalize $A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ if possible.

$$0 = \det(A - \lambda I) = \cdots = -(\lambda - 1)(\lambda + 2)^2$$

$\lambda_1 = 1$, $\lambda_2 = -2$

mult. 1   mult. 2

$$(A - 1I)\vec{x} = \vec{0} \to \left(\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \rightsquigarrow \text{eigsp.} = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$(A - (-2)I)\vec{x} = \vec{0} \to \left(\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right) \rightsquigarrow \text{eigsp.} = \operatorname{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

every eigvect is a multiple of either $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

cannot find 3 lin. indep. eigvects.

So $\boxed{A \text{ is } \underline{\text{not}} \text{ diagonalizable}}$

<u>Thm 6</u> If A is $n\times n$ and has n distinct eig. vals, then A is diagonalizable.

<u>Pf idea:</u> Use Thm 2 (eig. vects corr. to distinct eig. vals. are lin. indep.)

<u>Expl 5</u>

Is $A = \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$ diagonalizable?

A is triangular, so eigvals are $5, 0, -2$

A is 3×3 w/ 3 distinct eigvals.

By Thm 6, $\boxed{A \text{ is diagonalizable}}$

A : $n \times n$

$\lambda_1, \ldots, \lambda_p$: all the distinct eig. vals of A $(p \leq n)$

E_k = eig. space corr. to λ_k $= \text{Nul}(A - \lambda_k I)$

$d_k = \dim(E_k)$

$\mathcal{B}_k$ = a basis for E_k

$p(\lambda)$ = char. poly. of A

m_k = multiplicity of λ_k

<u>Thm 7(a)</u> $1 \leq d_k \leq m_k$ for all k

<u>Thm 7(b)</u> TFAE:

- A is diagonalizable
- $d_1 + d_2 + \cdots + d_p = n$
- $d_k = m_k$ for all k, and

  $p(\lambda) = (\lambda_1 - \lambda)^{m_1} (\lambda_2 - \lambda)^{m_2} \cdots (\lambda_p - \lambda)^{m_p}$

  only linear factors allowed here

<u>Thm 7(c)</u>

If A is diagonalizable, then

$\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Expl 6

Diagonalize $A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{pmatrix}$ if possible.

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 0 & 0 \\ 1 & 4 & -3-\lambda & 0 \\ -1 & -2 & 0 & -3-\lambda \end{vmatrix}$$

triangular

$$\overset{\downarrow}{=} (5-\lambda)^2 (-3-\lambda)^2$$

$$= (\lambda - 5)^2 (\lambda + 3)^2 = 0$$

eigvals: $\lambda_1 = 5$, $\lambda_2 = -3$

mult. 2 mult. 2

E_1 = eigenspace corr. to $\lambda_1 = 5$

$\dim E_1 \leq 2$ ← mult. of $\lambda_1 = 5$

so $\dim E_1$ is either 1 or 2

$\mathcal{B}_1$ = basis for E_1 = ?

$$(A - \lambda_1 I)\vec{x} = \vec{0}$$

$$(A - 5I)\vec{x} = \vec{0}$$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \end{array}\right) \begin{array}{c} R_4 + R_3 \to R_4 \\ R_1 \leftrightarrow R_3 \\ R_2 \leftrightarrow 4 \end{array} \left(\begin{array}{cccc|c} 1 & 4 & -8 & 0 & 0 \\ 0 & 2 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} \frac{1}{2}R_2 \to R_2 \\ R_1 - 4R_2 \to R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 8 & 16 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -8x_3 - 16x_4 \\ 4x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -8 \\ 4 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{pmatrix} -16 \\ 4 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_2}$$

$\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2\}$, $\dim E_1 = 2$

E_2 = eigenspace corr. to $\lambda_2 = -3$

$\dim E_2 \leq 2$ ← mult. of $\lambda_2 = -3$

so $\dim E_2$ is either 1 or 2

$\mathcal{B}_2$ = basis for E_2 = ?

$$(A - \lambda_2 I)\vec{x} = \vec{0}$$

$$(A + 3I)\vec{x} = \vec{0}$$

$$\left(\begin{array}{cccc|c} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \frac{1}{8}R_1 \to R_1 \\ R_3 - R_1 \to R_3 \\ R_4 + R_1 \to R_4 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} \frac{1}{8}R_2 \to R_2 \\ R_3 - 4R_2 \to R_3 \\ R_4 + 2R_2 \to R_4 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_3} + x_4 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_4}$$

$\mathcal{B}_2 = \{\vec{v}_3, \vec{v}_4\}$, $\dim E_2 = 2$

$\dim E_1 + \dim E_2 = 4$, A is 4×4, so

A is diagonalizable and

$\mathcal{B}_1 \cup \mathcal{B}_2 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is an eigvect. basis for $\mathbb{R}^4$

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix} = \begin{pmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

match

$$D = \begin{pmatrix} \lambda_1\vec{e}_1 & \lambda_2\vec{e}_2 & \lambda_3\vec{e}_3 & \lambda_4\vec{e}_4 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$A = PDP^{-1}$$