Test 2 review   Sections: 3.1-3.2, 4.1-4.6, 5.1-5.3

Determinants

Cofactor expansion $\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

A triangular $\Rightarrow$ $\det A = a_{11} a_{22} \cdots a_{nn}$

row operations

$A \xrightarrow[\text{replacement}]{} B$ $\det A = \det B$

$A \xrightarrow[\text{interchange}]{} B$ $\det A = -\det B$

$A \xrightarrow[\substack{\text{scaling} \\ cR_i \to R_i}]{} B$ $\det A = \frac{1}{c} \det B$

$\det A = 0 \longleftrightarrow$ A is noninvertible (singular)

$\det A \neq 0 \longleftrightarrow$ A is invertible (nonsingular)

$\det A = 0 \longleftrightarrow$ cols of A lin. dep.

$\det A = 0 \longleftrightarrow$ rows of A lin. dep.

Vector spaces

Definition involves 10 axioms
Scalars are real numbers

Expls

$\mathbb{R}^n$: Euclidean space

$\mathbb{R}^A$: space of functions from A to $\mathbb{R}$

$\mathbb{R}^{\mathbb{N}}$: space of sequences

$\mathbb{R}^{\mathbb{Z}}$: space of doubly-infinite sequences

$\mathbb{P}$: space of polynomials

$\mathbb{P}_n$: space of polynomials of degree $\leq n$

$C([a,b])$: space of continuous functions on $[a,b]$

$C^k([a,b])$: space of functions on $[a,b]$ with a continuous k^{th} derivative subspace : a subset that's also a vector space

H is a subspace of V iff

- $H \subset V$ ($H = V$ is okay)
- $\vec{0} \in H$
- H is closed under addition and scalar multiplication

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace

for $A = (\vec{a}_1 \cdots \vec{a}_n) = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{pmatrix}$:

$$\left.\begin{array}{ll} \text{nullspace of } A: & \text{Nul}\, A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\} \subset \mathbb{R}^n \\ \text{columnspace of } A: & \text{Col}\, A = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} \subset \mathbb{R}^m \\ \text{row space of } A: & \text{Row}\, A = \text{Span}\{\vec{r}_1, \dots, \vec{r}_m\} \subset \mathbb{R}^n \end{array}\right\}$$

all are subspaces

$\text{Row}\, A = \text{Col}\, A^T$

If $T: V \to W$ is linear,

the kernel of T is $\{\vec{v} \in V : T(\vec{v}) = \vec{0}\} \subset V$

aka the nullspace of T

subspaces

the range of T is $\{T(\vec{v}) : \vec{v} \in V\} \subset W$

$\mathcal{B}$ is a basis for V if

- $\mathcal{B}$ is lin. indep. (therefore finite)
- $\operatorname{Span}\mathcal{B} = V$

$\{\vec{e}_1, \ldots, \vec{e}_n\}$: standard basis for $\mathbb{R}^n$

$\{1, t, t^2, \ldots, t^n\}$: standard basis for $\mathbb{P}_n$

<u>Spanning set theorem</u>

$H = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

One at a time, throw away vectors that are linear combos of the others until what remains is lin. indep.

Then you'll have a basis for H.

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for A $m \times n$:

<u>Finding a basis for Nul A</u>

$$(A \mid \vec{0}) \sim \left(\begin{array}{c|c} \text{reduced} & \\ \text{echelon} & \vec{0} \\ \text{form} & \end{array}\right)$$

write soln set in parametric form

e.g. $\vec{x} = x_3 \begin{pmatrix} \vdots \end{pmatrix} + x_4 \begin{pmatrix} \vdots \end{pmatrix} + x_7 \begin{pmatrix} \vdots \end{pmatrix}$

free vars

basis for Nul A

Finding a basis for Col A

$A \sim \begin{pmatrix} \text{echelon} \\ \text{form} \end{pmatrix}$ ← identify the pivot cols

pivot cols of $\underline{\underline{A}}$ are a basis for Col A

In both cases, dim is # of pivots, so dim Col A = dim Row A

Finding a basis for Row A

$A \sim \begin{pmatrix} \text{echelon} \\ \text{form} \end{pmatrix}$

A and its echelon form have same row space; but must use rows of echelon form, not A

nonzero rows of $\underline{\underline{\text{echelon form}}}$ are a basis for Row A

V a vector sp., $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ a basis for V

$$(\vec{x})_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \longleftrightarrow \vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

$\mathcal{B}$ - coordinate vector of $\vec{x}$

$\vec{x} \longmapsto (\vec{x})_{\mathcal{B}}$: coordinate mapping determined by $\mathcal{B}$ ← it's an isomorphism: linear, one-to-one, and onto

$\vec{x}$ in V, $(\vec{x})_{\mathcal{B}}$ in $\mathbb{R}^n$

If V has a basis, then

- all bases have the same # of vectors
- $\dim V$ = # of vectors in a basis
- V is finite-dimensional

If V doesn't have a basis, then

- V is infinite-dimensional

Exception:

- $\{0\}$ has no basis (no lin. indep. subset)
- $\{0\}$ is finite-dimensional
- $\dim \{0\} = 0$

If $\dim V = n$, $p > n$, and $S = \{\vec{v}_1, \dots, \vec{v}_p\}$, then S is lin. dep.

If $\operatorname{Span} S = V$, then S can be reduced to a basis.

If S is lin. indep., then S can be expanded to a basis.

If $\dim V = n$ and $S = \{\vec{v}_1, \dots, \vec{v}_n\}$, then

- S lin. indep. $\Rightarrow$ S is a basis
- $\operatorname{Span} S = V$ $\Rightarrow$ S is a basis

for A $m \times n$:

$\operatorname{rank} A = \dim \operatorname{Col} A = \#$ of pivot cols

$\operatorname{nullity} A = \dim \operatorname{Nul} A = \#$ of free vars

$\operatorname{rank} A + \operatorname{nullity} A = \#$ of cols in A ← Rank Thm

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V a vector sp., $\mathcal{B}, \mathcal{C}$ bases, $\dim V = n$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left((\vec{b}_1)_{\mathcal{C}} \cdots (\vec{b}_n)_{\mathcal{C}} \right)$$ ← change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ (an $n \times n$ matrix)

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} (\vec{x})_{\mathcal{B}} = (\vec{x})_{\mathcal{C}}$$

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$$

$$\underset{\mathcal{D} \leftarrow \mathcal{C}}{P} \; \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \underset{\mathcal{D} \leftarrow \mathcal{B}}{P}$$

If $V = \mathbb{R}^n$ and $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$, then

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = (\vec{b}_1 \cdots \vec{b}_n)$$ ← also called $P_{\mathcal{B}}$

Finding $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ when $V = \mathbb{R}^n$:

$$(\vec{c}_1 \cdots \vec{c}_n \mid \vec{b}_1 \cdots \vec{b}_n) \sim \left(I_n \;\middle|\; \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \right)$$

for A $n \times n$:

$\lambda \in \mathbb{R}$ an eigval of A $\longleftrightarrow$ $A\vec{x} = \lambda \vec{x}$ for some $\vec{x} \neq \vec{0}$

$\longleftrightarrow$ $\underbrace{\det(A - \lambda I)}_{\substack{\text{characteristic polynomial} \\ \text{(degree } n)}} = 0$

$\vec{x} \in \mathbb{R}^n$ an eigvect of A $\longleftrightarrow$ $\vec{x} \neq \vec{0}$ and $\begin{array}{c} A\vec{x} = \lambda\vec{x} \\ \Updownarrow \\ (A - \lambda I)\vec{x} = \vec{0} \end{array}$

eigenspace of A corr. to λ = $\{\vec{0}\} \cup \{\text{eigenvectors}\}$

If A is triangular, eigvals are $a_{11}, a_{22}, \ldots, a_{nn}$

eigvects corr. to distinct eigvals are lin. indep.

i.e. $\lambda_1, \lambda_2, \ldots, \lambda_r$ distinct eigvals

$\updownarrow \quad \updownarrow \qquad \updownarrow$

$\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r}_{\text{lin. indep.}}$ corr. eigvects

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<u>Difference eqn:</u> $\vec{x}_{k+1} = A\vec{x}_k$

If λ is an eigval of A and
$\vec{x}_0$ is an eigvect corr. to λ,

then $\vec{x}_k = \lambda^k \vec{x}_0$ is a soln to the diff. eqn.

The multiplicity of an eigval is its multiplicity as a root of the characteristic polynomial.

A and B are similar if $P^{-1}AP = B$ for some P.

- Similar matrices have the same char. poly.
- Row equiv. matrices are <u>not</u> similar, in general.

A is diagonalizable if A is similar to a diagonal matrix D.

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diagonal

Diagonalizing a matrix A means finding P and D such that $A = PDP^{-1}$

$$A = \overbrace{\begin{pmatrix} | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{pmatrix}}^{P} \overbrace{\begin{pmatrix} d_{11} & & 0 \\ & d_{22} & \\ 0 & & \ddots \quad d_{nn} \end{pmatrix}}^{D} \overbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}^{P^{-1}}$$

cols are lin. indep. eigvects of A (an eigenbasis of $\mathbb{R}^n$)

entries are eigvals of A, they match the cols of P from left to right

not necessarily distinct

Diagonalizing a matrix (A is $n \times n$)

Find the distinct eigvals:

$$\lambda_1, \lambda_2, \ldots, \lambda_p \qquad (p \le n)$$

Find their multiplicities:

$$m_1, m_2, \ldots, m_p$$

If $m_1 + \cdots + m_p < n$, then A is not diagonalizable.

For each λ_i, find the eigsp E_i

- if $\dim E_i < m_i$, then A is not diagonalizable
- find a basis $\mathcal{B}_i$ for E_i

an eigvect basis for $\mathbb{R}^n$

Put the bases together:

$$\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_p = \mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$$

$$P = (\vec{b}_1 \cdots \vec{b}_n), \quad D = \begin{pmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{pmatrix}$$

d_{ii} is the eigval corr. to λ_i