5.4 Supplement

$(\Omega, \mathcal{F}, P)$: prob. sp.

$X_1, X_2, X_3, \dots$ real-val. rvs

$S_n = X_1 + X_2 + \cdots + X_n$

$\{X_1, X_2, \dots\}$ are <u>independent</u> if, for all $n \in \mathbb{N}$ and all $B_j \in \mathcal{R}$,

$$P\left(\bigcap_{j=1}^{n} \{X_j \in B_j\} \right) = \prod_{j=1}^{n} P(X_j \in B_j)$$

They are <u>i.i.d.</u> if they are independent and they all have the same distribution.

<u>Thm (LLN)</u> (will see later)

If $X_1, X_2, \dots$ are i.i.d. and $X_1 \in L^1(\Omega, \mathcal{F}, P)$, then

$$\frac{S_n}{n} \xrightarrow{n \to \infty} \underbrace{\int_\Omega X_1 \, dP}_{=: E[X_1]} \quad \text{a.s.}$$

("expected value")

Special case:

$X_n = 1_{A_n}$

Then $S_n =$ # of events in $A_1, \dots, A_n$ that are true

and $\int_\Omega X_1 \, dP = \int_\Omega 1_{A_1} \, dP = P(A_1)$

will define later

So if $A_1, A_2, \dots$ are independent

and $P(A_n)$ does not depend on n,

then $\frac{S_n}{n} \xrightarrow{n \to \infty} P(A_1)$ a.s.

proportion of $A_1, \dots, A_n$ that are true

Expected value should be interpreted in the context of this theorem (and others like it)

(will see later)

Thm (CLT)

Suppose $X_1, X_2, \ldots$ are i.i.d and $X_1 \in L^2(\Omega, \mathcal{F}, P)$. Let $Y_n = \sqrt{n}\left(\frac{S_n}{n} - E[X_1]\right)$, and let F_n be the distribution function of Y_n. Let $Z \sim N(0,1)$, and, for $\sigma > 0$, let F_σ be the distribution function of σZ. Then $F_n \to F_\sigma$ pointwise,

where

$$\sigma^2 = \underbrace{\int_\Omega (X_1 - E[X_1])^2 \, dP}_{=: \operatorname{var}(X)}$$

"variance"

Variance should be interpreted in the context of this theorem (and others like it)

Expected value for simple rvs

$\{A_1, \dots, A_n\} \subset \mathcal{F}$ a partition of Ω

$c_1, \dots, c_n \in \mathbb{R}$

$X = \sum_{j=1}^n c_j \mathbb{1}_{A_j}$ (simple rv)

- A_j's are events, or "sentences"
- exactly one of them is true
- $X_j = c_j$ in the case that A_j is true

$$E[X] = \int_\Omega X \, dP = \int_\Omega \sum_{j=1}^n c_j \mathbb{1}_{A_j} \, dP$$

$$= \sum_{j=1}^n c_j P(A_j) = \sum_{j=1}^n c_j P(X = c_j)$$

$E[X]$ is a weighted average of $\underbrace{c_1, \dots, c_n}_{\text{the possible values of } X}$,

where c_j gets weight $P(X = c_j)$.