Ch. 4 continued

Let $(\Omega, \mathcal{F}, P)$ be a prob. space.
The set Ω is called the sample space.
Elements $\omega \in \Omega$ are called outcomes.
Set $A \in \mathcal{F}$ are called events.

A random variable is a m'ble function $X : \Omega \to S$, where $(S, \mathcal{S})$ is a m'ble space.
The set S is called the state space of X.

A (measure-theoretic) probability model is a probability space, $(\Omega, \mathcal{F}, P)$, together with a family of random variables, $\{X_i : i \in I\}$, where X_i takes values in a m'ble space, $(S_i, \mathcal{S}_i)$.

An outcome ω can be thought of as a "structure" or an "interpretation". In ω, we can interpret sentences that use the "symbols" X_i.

Expl

$\Omega = \{-1, 1\} \times [0,1]$

$\mathcal{F} = 2^{\{0,1\}} \otimes \mathcal{B}_{[0,1]}$

$P = \mu \times \lambda$, where $\mu(\{-1\}) = \mu(\{1\}) = \frac{1}{2}$

Each $\omega \in \Omega$ has the form $\omega = (\omega_0, \omega_1)$,

where $\omega_0 = -1$ or $\omega_0 = 1$, and $\omega_1 \in [0,1]$.

Define mble $X : \Omega \to \mathbb{R}$ by $X(\omega) = \omega_1$

and mble $Y : \Omega \to \mathbb{R}$ by $Y(\omega) = \omega_0 \omega_1$

Then X and Y are $(\mathbb{R}, \mathcal{R})$-valued random vars.

$(\Omega, \mathcal{F}, P)$, X, Y constitute a probability model.

Fix $\omega \in \Omega$. In ω, we can interpret sentences that use the symbols "X" and "Y". For example, the sentence "$X = Y$" is interpreted as "$X(\omega) = Y(\omega)$".

If $\omega = (1, 0.6)$, then "$X = Y$" is interpreted as "$0.6 = 0.6$", which is true.

If $\omega = (-1, \frac{1}{3})$, then "$X = Y$" is interpreted as "$\frac{1}{3} = -\frac{1}{3}$", which is false.

We can identify the sentence "$X = Y$" with the set of interpretations in which it is true:

$$\text{"}X = Y\text{"} \longleftrightarrow \{\omega \in \Omega : X(\omega) = Y(\omega)\}.$$

in the "real" world, we assign probabilities to sentences

We then say the probability that $X = Y$ is

$$P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}),$$

in measure-theory, we assign probabilities to sets

which we write in shorthand as $P(X = Y)$.

Logic vs. set theory

sentences	events

operations

$\varphi =$ "$X = Y$" $\qquad A_\varphi = \{\omega \in \Omega : X(\omega) = Y(\omega)\}$

$\psi =$ "$Y + 1 > X$" $\qquad A_\psi = \{\omega \in \Omega : Y(\omega) + 1 > X(\omega)\}$

not

$\neg\varphi =$ "$X \neq Y$" $\qquad A_\varphi^c$

or

$\varphi \vee \psi \qquad A_\varphi \cup A_\psi$

and

$\varphi \wedge \psi \qquad A_\varphi \cap A_\psi$

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relations

$\varphi \vdash \psi \qquad A_\varphi \subset A_\psi$

logically implies

$X(\omega)$ can be thought of as the "meaning" of X in the "interpretation" ω.

Or in other language:

The sample space Ω consists of many possible states of the world.

The value of the symbol "X" in the state of the world ω is $X(\omega)$.

In logic, "X" would be a <u>constant symbol</u>.

So "random variable" is maybe not the best name for it. A better name might be "uncertain constant".

Where did $(\Omega, \mathcal{F}, P)$, X, Y come from in that example? I wanted these statements to be true:

$$P(X \in [0,1]) = 1$$

$$P(X \in [a,b]) = b - a \quad \text{for } [a,b] \subset [0,1]$$

$$P(Y = X \mid X) = P(Y = -X \underbrace{\mid X}_{\text{will learn about this later}}) = \frac{1}{2}$$

$(\Omega, \mathcal{F}, P)$, X, Y was constructed to make this happen.