5.2 The characteristic eqn

Expl 1

Find the eigenvals. of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

λ is an eigval of A iff $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-6-\lambda) - 9 \\ &= (\lambda-2)(\lambda+6) - 9 \\ &= \lambda^2 + 4\lambda - 12 - 9 \\ &= \lambda^2 + 4\lambda - 21 \end{aligned}$$

Need to solve

$$\underbrace{\lambda^2 + 4\lambda - 21}_{\text{characteristic polynomial}} = 0 \leftarrow \text{characteristic equation of } A$$

$$(\lambda + 7)(\lambda - 3) = 0$$

$\boxed{\lambda = -7 \text{ and } \lambda = 3 \text{ are the eigenvalues}}$

Expl 3

Find the characteristic eqn of

$$A = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$ ← triangular

$$\det(A - \lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$
$$= (-1)^4 (\lambda-5)(\lambda-3)(\lambda-5)(\lambda-1)$$
$$= (\lambda-5)^2(\lambda-3)(\lambda-1)$$

$$\boxed{\underbrace{(\lambda-5)^2(\lambda-3)(\lambda-1)}_{\text{characteristic polynomial}} = 0}$$ ← characteristic equation

In Expl 3, eig. vals. are

$\lambda = 5$ ⟵ multiplicity 2

$\lambda = 3$ ⟵ " 1

$\lambda = 1$ ⟵ " 1

The <u>multiplicity</u> of an eig. val. is the # of times it appears as a root of the char. eqn.

If A is $n \times n$, its char. poly. is a poly. of degree n.

<u>Expl 4</u>

A is 6×6 and has char. poly. $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eig. vals. of A and their multiplicities.

$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = 0$$
$$\lambda^4(\lambda^2 - 4\lambda - 12) = 0$$
$$\lambda^4(\lambda+2)(\lambda-6) = 0$$

$\lambda = 0$ mult. 4
$\lambda = -2$ mult. 1
$\lambda = 6$ mult. 1

If A and B are $n \times n$, then A and B are <u>similar</u> if there exists P such that

$$\underbrace{P^{-1}AP = B}_{\text{"similarity transformation"}} \longrightarrow$$

equiv. to $A = PBP^{-1}$

Thm 4

If A and B are similar, then they have the same char. poly.

Pf:
$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}(AP - \lambda P)$$
$$= P^{-1}(A - \lambda I)P.$$

$$\therefore \quad \det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P)$$
$$= \det(P^{-1})\det(A - \lambda I)\det(P)$$
$$= \det(A - \lambda I)\det(P^{-1})\det(P)$$
$$= \det(A - \lambda I)\det(P^{-1}P)$$
$$= \det(A - \lambda I)\cancel{\det(I)}\ 1$$
$$= \det(A - \lambda I). \quad \square$$

Warnings:

- Converse not true.

  Same char. poly. $\not\Rightarrow$ similar

  e.g. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ have same char. poly, but <u>not</u> similar

- similarity $\neq$ row equivalence.

  row operations usually change the eig. vals.

Expl 5

$$A = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\vec{x}_{k+1} = A\vec{x}_k.$$

Find an explicit expression for $\vec{x}_k$ and find $\lim_{k\to\infty} \vec{x}_k$ if it exists.

Find eigvals:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 0.95-\lambda & 0.03 \\ 0.05 & 0.97-\lambda \end{vmatrix}$$

$$= (0.95-\lambda)(0.97-\lambda) - 0.0015$$

$$= \lambda^2 - 1.92\lambda + (0.95)(0.97) - 0.0015$$

$$= \lambda^2 - 1.92\lambda + (1-0.05)(1-0.03) - 0.0015$$

$$= \lambda^2 - 1.92\lambda + 1 - 0.08 + 0.0015 - 0.0015$$

or use a calculator

$$= \lambda^2 - 1.92\lambda + 0.92$$

$\lambda = 1$ is a root

$$\lambda^2 - 1.92\lambda + 0.92 = (\lambda - 1)(\lambda - 0.92) = 0$$

eigvals: $\lambda_1 = 1, \ \lambda_2 = 0.92 \leftarrow$ or use quadratic equation

Find an eigvect. for $\lambda_1 = 1$:

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$\left(\begin{array}{cc|c} -0.05 & 0.03 & 0 \\ 0.05 & -0.03 & 0 \end{array}\right) \begin{array}{l} R_2 + R_1 \to R_2 \\ -\frac{1}{0.05} R_1 \to R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} \frac{3}{5} \\ 1 \end{pmatrix}$$

Take $x_2 = 5$: $\quad \vec{v}_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Find an eigvect. for $\lambda_2 = 0.92$

$$(A - 0.92 I)\vec{v}_2 = \vec{0}$$

$$\left(\begin{array}{cc|c} 0.03 & 0.03 & 0 \\ 0.05 & 0.05 & 0 \end{array}\right) \begin{array}{l} R_2 - \frac{5}{3} R_1 \to R_2 \\ \frac{1}{0.03} R_1 \to R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Take $x_2 = -1$: $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_1 = 1$ $\qquad$ $\lambda_2 = 0.92$

$\vec{v}_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ $\qquad$ $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

lin. indep.
so a basis for $\mathbb{R}^2$

$$\vec{x}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Find c_1, c_2

$$\begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$= \frac{1}{-3-5} \begin{pmatrix} -1 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$$

$$= -\frac{1}{8} \begin{pmatrix} -1 \\ -3 + \frac{6}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ -\frac{1}{8}\left(-\frac{9}{5}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ \frac{9}{40} \end{pmatrix}$$

$$= \begin{pmatrix} 0.125 \\ 0.225 \end{pmatrix}$$

$$\vec{x}_0 = 0.125\,\vec{v}_1 + 0.225\,\vec{v}_2$$

$$\vec{x}_1 = A\vec{x}_0 = 0.125\,A\vec{v}_1 + 0.225\,A\vec{v}_2$$

$$= 0.125\,\lambda_1\vec{v}_1 + 0.225\,\lambda_2\vec{v}_2$$

$$= 0.125\,\vec{v}_1 + 0.225(0.92)\,\vec{v}_2$$

$$\vec{x}_2 = A\vec{x}_1 = 0.125\,\underbrace{\cancel{A\vec{v}_1}}_{\vec{v}_1} + 0.225(0.92)\,\underbrace{\cancel{A\vec{v}_2}}_{0.92\vec{v}_2}$$

$$= 0.125\,\vec{v}_1 + 0.225(0.92)^2\,\vec{v}_2$$

this exponent keeps increasing

$$\boxed{\vec{x}_k = 0.125\begin{pmatrix} 3 \\ 5 \end{pmatrix} + 0.225(0.92)^k\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\lim_{k\to\infty} (0.92)^k = 0$$

So $\boxed{\lim_{k\to\infty} \vec{x}_k = 0.125\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}}$