5.1 Eigenvalues and eigenvectors

A : an $n \times n$ matrix

If $\lambda \in \mathbb{R}$ and $A\vec{x} = \lambda \vec{x}$ has a nontrivial solution, then λ is an <u>eigenvalue</u> of A.

Any nontrivial soln is an <u>eigenvector</u> of A corresponding to λ.

The set of all solutions is the <u>eigenspace</u> of A corresponding to λ.

$$
\begin{aligned}
A\vec{x} = \lambda \vec{x} \quad & \text{iff} \quad A\vec{x} - \lambda \vec{x} = \vec{0} \\
& \text{iff} \quad A\vec{x} - \lambda I \vec{x} = \vec{0} \\
& \text{iff} \quad (A - \lambda I)\vec{x} = \vec{0}
\end{aligned}
$$

$\therefore$ λ is an eigenvalue iff $(A - \lambda I)\vec{x} = \vec{0}$ has nontrivial solns

iff $A - \lambda I$ is singular

iff $\det(A - \lambda I) = 0$.

eigenspace $= \mathrm{Nul}(A - \lambda I)$

So eigenspace is a subspace of $\mathbb{R}^n$.

- eigenvalue can be 0
- eigenvector cannot be $\vec{0}$
- eigenspace $= \{\vec{0}\} \cup \{\text{all eigenvectors}\}$

Expl 2

$$A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(a) Is $\vec{u}$ an eigenvector of A?
(b) Is $\vec{v}$ " ?
(c) Is $\vec{w}$ " ?

(a) Check if $A\vec{u} = \lambda \vec{u}$ for some λ:

$$A\vec{u} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} \stackrel{?}{=} \lambda \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

$\lambda = -4$ works $\boxed{\text{YES}}$

(b) $$A\vec{v} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \stackrel{?}{=} \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

No, not multiples. $\boxed{\text{No}}$

(c) $\vec{w} = \vec{0}$, so cannot be an eigvect. $\boxed{\text{NO}}$

Expl 3

$A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$. Is $\lambda = 7$ an eigenvalue?

If so, describe the set of eigenvectors corresponding to $\lambda = 7$. Also describe the eigenspace.

$\lambda = 7$ is an eigval. iff $A - 7I$ is singular

$$A - 7I = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}$$

rows are lin. dep. (multiples)

so $A - 7I$ is singular

$\boxed{\lambda = 7 \text{ is an eigenvalue}}$

eigenspace $= \mathrm{Nul}(A - 7I)$

$$\left(\underbrace{\begin{matrix} -6 & 6 \\ 5 & -5 \end{matrix}}_{A-7I} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix}\right) \sim \left(\begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix}\right)$$

$x_1 - x_2 = 0 \longrightarrow x_1 = x_2$

x_2 free

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\text{eigspace} = \left\{ t\begin{pmatrix}1\\1\end{pmatrix} : t \in \mathbb{R} \right\}}$$

$$\boxed{\text{set of eigvects} = \left\{ t\begin{pmatrix}1\\1\end{pmatrix} : t \in \mathbb{R},\ t \neq 0 \right\}}$$

<u>Expl 4</u>

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}. \quad \lambda = 2 \text{ is an eigenvalue.}$$

Find a basis for the eigenspace corresponding to $\lambda = 2$.

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eigspace $= \mathrm{Nul}(A - 2I)$

$$(A - 2I \mid \vec{0}) = \left(\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Soln set $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix}$

$$= x_2 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{basis} = \boxed{\left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

Thm 1

The eigenvalues of a triangular matrix are the entries on the main diagonal.

Pf: Let A be $n \times n$ and triangular. A number $\lambda \in \mathbb{R}$ is an eig. val. iff $\det(A - \lambda I) = 0$. But $A - \lambda I$ is also triangular with diagonal entries $a_{ii} - \lambda$. Thus, λ is an eig. val. iff

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda). \end{aligned}$$

The solus of this eqn are $a_{11}, a_{22}, \dots, a_{nn}$. $\square$

Expls:

$\begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$ eig. vals: $3, 0, 2$

$\begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{pmatrix}$ eig. vals: $4, 1$

Thm 2 Let A be $n \times n$. Let $\lambda_1, \dots, \lambda_r$ be distinct eig. vals. For each i between 1 and r, let $\vec{v}_i$ be an eig. vect. corresponding to λ_i.
Then $\{\vec{v}_1, \dots, \vec{v}_r\}$ are lin. indep.

Pf: Assume $\{\vec{v}_1, \dots, \vec{v}_r\}$ are lin. dep.

Since $\vec{v}_1$ is an eig. vect., $\vec{v}_1 \neq \vec{0}$.

Thus, $\vec{v}_j = c_1 \vec{v}_1 + \cdots + c_{j-1} \vec{v}_{j-1}$ for some j.

Let j_0 be the smallest such j and define $p = j_0 - 1$. Then

- $\vec{v}_{p+1} = c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p$, and
- $\{\vec{v}_1, \dots, \vec{v}_p\}$ are lin. indep.

Now,

$$\begin{aligned} \lambda_{p+1} \vec{v}_{p+1} &= A \vec{v}_{p+1} \\ &= A(c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p) \\ &= c_1 A \vec{v}_1 + \cdots + c_p A \vec{v}_p \\ &= c_1 \lambda_1 \vec{v}_1 + \cdots + c_p \lambda_p \vec{v}_p \end{aligned}$$

Thus,

$$\vec{0} = c_1 \lambda_1 \vec{v}_1 + \cdots + c_p \lambda_p \vec{v}_p - \lambda_{p+1} \vec{v}_{p+1}$$

$$= c_1 \lambda_1 \vec{v}_1 + \cdots + c_p \lambda_p \vec{v}_p - \lambda_{p+1}(c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p)$$

$$= c_1(\lambda_1 - \lambda_{p+1}) \vec{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1}) \vec{v}_p .$$

But $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are lin. indep.

So, for every i between 1 and p,

$$c_i (\lambda_i - \lambda_{p+1}) = 0.$$

But $\lambda_i \neq \lambda_{p+1}$, so $c_i = 0$ for every i.

Thus,

$$\vec{v}_{p+1} = c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p = \vec{0},$$

which is impossible, since $\vec{v}_{p+1}$ is an eig. vect. $\square$

Difference eqns

$y(t)$: position of a mass on a spring at time t

$$my'' + by' + ky = 0$$

mass

damping constant (friction)

spring constant

$$\underline{t_1 = t_0 + \Delta t}$$

$$y_0 = y(t_0) \qquad y_1 = y(t_1)$$

$$v_0 = y'(t_0) \qquad v_1 = y'(t_1)$$

$$a_0 = y''(t_0) \qquad a_1 = y''(t_1)$$

given these ... how to compute these?

$$y_1 = y(t_1) = y(t_0 + \Delta t) \approx \underbrace{y(t_0)}_{y_0} + \underbrace{y'(t_0)}_{v_0}\Delta t$$

linear approximation

$$y_1 \approx y_0 + v_0 \Delta t \checkmark$$

$$v_1 = y'(t_1) = y'(t_0 + \Delta t) \approx y'(t_0) + y''(t_0)\Delta t$$

$$v_1 \approx v_0 + a_0 \Delta t \checkmark$$

$$a_1 \approx ?$$

$$my'' + by' + ky = 0$$

$$my''(t_1) + by'(t_1) + ky(t_1) = 0$$

$$ma_1 + bv_1 + ky_1 = 0$$

$$a_1 = -\frac{b}{m}v_1 - \frac{k}{m}y_1$$

$$\approx -\frac{b}{m}(v_0 + a_0\Delta t) - \frac{k}{m}(y_0 + v_0 \Delta t)$$

$$= -\frac{k}{m}y_0 - \frac{b + k\Delta t}{m}v_0 - \frac{b\Delta t}{m}a_0 \checkmark$$

Put all 3 eqns together ...

$$\underbrace{\begin{pmatrix} y_1 \\ v_1 \\ a_1 \end{pmatrix}}_{\vec{x}_1} = \underbrace{\begin{pmatrix} 1 & \Delta t & 0 \\ 0 & 1 & \Delta t \\ -\frac{k}{m} & -\frac{b+k\Delta t}{m} & -\frac{b\Delta t}{m} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_0 \\ v_0 \\ a_0 \end{pmatrix}}_{\vec{x}_0}$$

Can continue: $\vec{x}_{k+1} = A\vec{x}_k$

recursively defines a sequence of vectors in $\mathbb{R}^3$ that approximates the functions y, y', y''.

If Δt is small,

$$\vec{x}_k = \begin{pmatrix} y_k \\ v_k \\ a_k \end{pmatrix} \approx \begin{pmatrix} y(t_0 + k\Delta t) \\ y'(t_0 + k\Delta t) \\ y''(t_0 + k\Delta t) \end{pmatrix}$$

We have turned the differential eqn into a difference eqn

How to solve for $\vec{x}_k$?

Let λ be an eig. val of A w/ eig. vector $\vec{x}_0$.

Then

$$\vec{x}_1 = A\vec{x}_0 = \lambda\vec{x}_0$$

$$\vec{x}_2 = A\vec{x}_1 = A(\lambda\vec{x}_0) = \lambda A\vec{x}_0 = \lambda(\lambda\vec{x}_0) = \lambda^2\vec{x}_0$$

$$\vec{x}_3 = A\vec{x}_2 = \lambda^2 A\vec{x}_0 = \lambda^3\vec{x}_0$$

$$\vdots$$

$$\vec{x}_k = \lambda^k\vec{x}_0 \longleftarrow \text{explicit soln}$$

$$\vdots$$

What to do if $\vec{x}_0$ not an eig. vect.?

Answer in Expl 5 of Section 5.2.