4.6 Change of basis

Expl 1

V : a vector space

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ a basis

$\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ another basis

$(\vec{x})_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$, $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$

Find $(\vec{x})_{\mathcal{C}}$.

$$(\vec{x})_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \vec{x} = 3\vec{b}_1 + \vec{b}_2$$

$$(\vec{x})_{\mathcal{C}} = (3\vec{b}_1 + \vec{b}_2)_{\mathcal{C}} = 3(\vec{b}_1)_{\mathcal{C}} + (\vec{b}_2)_{\mathcal{C}}$$

$$= \Big(\underset{?}{\underset{\uparrow}{(\vec{b}_1)_{\mathcal{C}}}} \quad \underset{?}{\underset{\uparrow}{(\vec{b}_2)_{\mathcal{C}}}} \Big) \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2 \Rightarrow (\vec{b}_1)_{\mathcal{C}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2 \Rightarrow (\vec{b}_2)_{\mathcal{C}} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$

$$(\vec{x})_{\mathcal{C}} = \begin{pmatrix} 4 & -6 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 6 \\ 4 \end{pmatrix}}$$

If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ are bases for a vector space V, we define the <u>change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$</u> by:

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left((\vec{b}_1)_{\mathcal{C}} \cdots (\vec{b}_n)_{\mathcal{C}} \right)$$

$n \times n$

- $\vec{b}_1, \dots, \vec{b}_n$ lin. indep.

  $\therefore$ $(\vec{b}_1)_{\mathcal{C}}, \dots, (\vec{b}_n)_{\mathcal{C}}$ lin. indep.

  $\therefore$ $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is invertible

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<u>Thm 15:</u> $\quad (\vec{x})_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} (\vec{x})_{\mathcal{B}}$

- Since $\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} (\vec{x})_{\mathcal{C}} = (\vec{x})_{\mathcal{B}}$, this tells us that $\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$.

$V = \mathbb{R}^n$

$\mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\}$ standard basis

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ another basis

$(\vec{b}_j)_{\mathcal{E}} = \vec{b}_j$

previous notation

$\therefore \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = (\vec{b}_1 \cdots \vec{b}_n) = P_{\mathcal{B}}$

Expl 2

$\vec{b}_1 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$, $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ basis for $\mathbb{R}^2$

$\vec{c}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\vec{c}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ "

Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left((\vec{b}_1)_{\mathcal{C}} \ (\vec{b}_2)_{\mathcal{C}} \right)$$

$(\vec{b}_1)_{\mathcal{C}} = ?$ $\quad \vec{b}_1 = x_1 \vec{c}_1 + x_2 \vec{c}_2$. Find x_1, x_2. Then $(\vec{b}_1)_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$(\vec{c}_1 \ \vec{c}_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{b}_1$. Solve w/ row reduction on $(\vec{c}_1 \ \vec{c}_2 \mid \vec{b}_1)$ Not yet!

$(\vec{b}_2)_{\mathcal{C}} = ?$ $\quad \vec{b}_2 = y_1 \vec{c}_1 + y_2 \vec{c}_2$. $\quad (\vec{b}_2)_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$(\vec{c}_1 \ \vec{c}_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \vec{b}_2$. Solve w/ row reduction on $(\vec{c}_1 \ \vec{c}_2 \mid \vec{b}_2)$

Combine the problems: work with

$$(\vec{c}_1 \ \vec{c}_2 \mid \vec{b}_1, \vec{b}_2)$$

$$\left(\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right)$$

$\uparrow \quad \uparrow$

lin. indep. and square
can reduce to I

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\xrightarrow[\text{reduction}]{\text{row}} \left(\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right)$$

$$(\vec{b}_1)_{\mathcal{C}} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \quad (\vec{b}_2)_{\mathcal{C}} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\boxed{\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}}$$

In general,

$$\boxed{(\vec{c}_1 \cdots \vec{c}_n \mid \vec{b}_1 \cdots \vec{b}_n) \sim (I \mid \underset{\mathcal{C} \leftarrow \mathcal{B}}{P})}$$