4.5 The dimension of a vector space

Thm 10

Suppose V is a vector sp. with basis $\{\vec{b}_1, \dots, \vec{b}_n\}$.
If $S \subset V$ has more than n vectors,
then S is lin. dep.

Pf sketch:

$S = \{\vec{u}_1, \dots, \vec{u}_p\}$, $p > n$

$$\underbrace{(\vec{u}_1)_{\mathcal{B}},\ (\vec{u}_2)_{\mathcal{B}},\ \dots,\ (\vec{u}_p)_{\mathcal{B}}}_{\text{lin. dep.}} \in \mathbb{R}^n, \quad p > n$$

$$c_1(\vec{u}_1)_{\mathcal{B}} + \cdots + c_p(\vec{u}_p)_{\mathcal{B}} = \vec{0} \leftarrow \text{in } \mathbb{R}^n$$

not all zero

$\vec{x} \mapsto (\vec{x})_{\mathcal{B}}$ is an isomorphism from V to $\mathbb{R}^n$. $\therefore$

$$c_1\vec{u} + \cdots + c_p\vec{u}_p = \vec{0} \leftarrow \text{in } V$$

So S in lin. dep. $\square$

Thm 11

Suppose V is a vector sp. with basis $\{\vec{b}_1, \dots, \vec{b}_n\}$.
Then every basis of V has n vectors.

Pf sketch:

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$.

Suppose $\mathcal{B}' = \{\vec{b}_1', \dots, \vec{b}_p'\}$ is another basis.

By Thm 10, $p > n \Rightarrow \mathcal{B}'$ lin. dep., but $\mathcal{B}'$ is a basis.
So $p \leq n$.

By Thm 10, $n > p \Rightarrow \mathcal{B}$ lin. dep., but $\mathcal{B}$ is a basis.
So $n \leq p$. $\therefore p = n$. $\square$

If V has a basis, then V is finite-dimensional and the number of elements in a basis is the dimension of V, which we write as $\dim V$.

If V has no basis, then V is infinite-dimensional.

Exception: $\{\vec{0}\}$ is finite-dimensional and $\dim \{\vec{0}\} = 0$.

Expl 1

$\mathbb{R}^n$ basis: $\{\vec{e}_1, \dots, \vec{e}_n\}$ $\dim \mathbb{R}^n = n$

$\mathbb{P}_2$ basis: $\{1, t, t^2\}$ $\dim \mathbb{P}_2 = 3$

$\mathbb{P}_n$ basis: $\{1, t, t^2, \dots, t^n\}$ $\dim \mathbb{P}_n = n+1$

$\mathbb{P}$ no basis $\dim \mathbb{P} = \infty$

Expl 3

$$H = \left\{ \begin{pmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}$$

Is H a subspace of $\mathbb{R}^4$? If so, find dim H.

$$\begin{pmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix}}_{\vec{v}_1} + b\underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2} + c\underbrace{\begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix}}_{\vec{v}_3} + d\underbrace{\begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix}}_{\vec{v}_4}$$

$H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, $\boxed{\text{H is a subspace}}$

$\vec{v}_1 \neq \vec{0}$ ✓

$\{\vec{v}_1, \vec{v}_2\}$ lin. indep. ✓

$\vec{v}_3 = 0\vec{v}_1 - 2\vec{v}_2$

discard it

$H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ (by Spanning set thm)

$\vec{v}_4 \overset{?}{=} c_1\vec{v}_1 + c_2\vec{v}_2$

No (look at last co-ordinate)

So $\vec{v}_1, \vec{v}_2, \vec{v}_4$ lin. indep.

basis for H: $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

$\boxed{\dim H = 3}$

Thm 12 counterpart to spanning set thm

Let V be a finite-dim. vector sp.

Let $H \subset V$ be a subspace.

- If $S \subset H$ is lin. indep., then H has a basis $\mathcal{B}$ with $S \subset \mathcal{B}$.
- H is finite-dim. and $\dim H \leq \dim V$.

Pf idea:

If $H = \text{Span}\, S$, then S is a basis.

If not, pick $\vec{v} \in H$, $\vec{v} \notin \text{Span}\, S$. Add $\vec{v}$ to S.

Repeat until it spans H. $\square$

Thm 13 (The basis thm)

Let V be a vector sp. with $\dim V = p$.
Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$.
If S is lin. indep., then S is a basis.
If S spans V, then S is a basis.

Pf idea

S lin. indep $\Rightarrow$ can enlarge to a basis
S lin. dep. $\Rightarrow$ can reduce to a basis
} but a basis must have p vectors. $\square$

A : an $m \times n$ matrix

The rank of A = $\dim \text{Col } A$ = # of pivot cols

(pivot cols of A are a basis for Col A)

The nullity of A = $\dim \text{Nul } A$ = # of free vars

(remember how to find basis by solving $A\vec{x} = \vec{0}$)

Thm 14 (The rank thm)

rank A + nullity A = # of columns in A.

Expl 5

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$ Find rank A and nullity A.

$$(A \mid \vec{0}) \xrightarrow[\text{reduction}]{\text{row}} \left(\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivots   free vars

$\boxed{\text{rank } A = 2}$ $\boxed{\text{nullity } A = 3}$

Expl 6a

A is 7×9, nullity $A = 2$. Find rank A.

$$\text{rank } A + \underbrace{\cancel{\text{nullity } A}}_{2} = 9 \leftarrow \text{\# of cols}$$

$$\boxed{\text{rank } A = 7}$$

Expl 6b

A is 6×9. Explain why nullity $A \neq 2$.

$A = (\vec{a}_1 \cdots \vec{a}_9)$, each $\vec{a}_j \in \mathbb{R}^6$

$\text{Col } A = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_9\} \subset \mathbb{R}^6$

$\text{rank } A = \dim \text{Col } A \leq \dim \mathbb{R}^6 = 6$

$\text{rank } A + \text{nullity } A = 9$

$\text{nullity } A = 9 - \text{rank } A \geq 9 - 6 = 3$. ✓

EXAMPLE 8 A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

$A\vec{x} = \vec{0}$, A is 40×42

$\vec{v}_1, \vec{v}_2$ are solns , lin. indep.

soln set $= \text{Span}\{\vec{v}_1, \vec{v}_2\}$

Does $A\vec{x} = \vec{b}$ have a soln for every $\vec{b}$?

$\text{Nul } A = \text{soln set of } A\vec{x} = \vec{0} = \text{Span}\{\underbrace{\vec{v}_1, \vec{v}_2}_{\text{lin. indep.}}\}$

$\text{nullity } A = \dim \text{Nul } A = 2$

$\text{rank } A + \text{nullity } A = \#\text{ of cols}$

$\text{rank } A + 2 = 42$

$\text{rank } A = 40$

Col A = span of cols of A

$= (\text{range of } T(\vec{x}) = A\vec{x}) \subset \mathbb{R}^{40}$

dim Col A = rank A = 40

So range of T = $\mathbb{R}^{40}$

$\boxed{\text{Yes, for every } \vec{b} \in \mathbb{R}^{40}\text{, there is a soln to } A\vec{x} = \vec{b}}$