# 4.4 Coordinate systems

$V$: a vector sp.

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ a basis for $V$

Every $\vec{x} \in V$ can be written as

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

These weights are unique.

Thm 8

They are called the $\mathcal{B}$-coordinates of $\vec{x}$

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Notation: $(\vec{x})_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ $\mathcal{B}$-coordinate vector of $\vec{x}$

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$V \longrightarrow \mathbb{R}^n$ # of basis vectors

$\vec{x} \longmapsto (\vec{x})_{\mathcal{B}}$

This function is the coordinate mapping (determined by $\mathcal{B}$)

It's linear, one-to-one, and onto (Thm 9)

an "isomorphism"

## Expl 1

$V = \mathbb{R}^2$, $\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathcal{B} = \underbrace{\{\vec{b}_1, \vec{b}_2\}}_{\text{a basis}}$

$(\vec{x})_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Find $\vec{x}$.

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$(\vec{x})_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ means $\vec{x} = -2\vec{b}_1 + 3\vec{b}_2$

$= -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 6 \end{pmatrix}}$

## Expl 2

$V = \mathbb{R}^2$, $\mathcal{B} = \underbrace{\{\vec{e}_1, \vec{e}_2\}}_{\text{standard basis}}$, $\vec{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. Find $(\vec{x})_{\mathcal{B}}$.

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$\vec{x} = 1\vec{e}_1 + 6\vec{e}_2 \implies (\vec{x})_{\mathcal{B}} = \boxed{\begin{pmatrix} 1 \\ 6 \end{pmatrix}}$

## Expl 4

$V = \mathbb{R}^2$, $\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$,

$\vec{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find $(\vec{x})_{\mathcal{B}}$.

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$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2$ , $(\vec{x})_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ ← need to find these

$\downarrow$

$$(\vec{b}_1 \ \vec{b}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{x}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \xrightarrow{\text{solve}} \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

$$(\vec{x})_{\mathcal{B}} = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ a basis for $\mathbb{R}^n$

$P_{\mathcal{B}} = (\vec{b}_1 \cdots \vec{b}_n)$ ← <u>change-of-coordinates matrix</u>

columns are lin. indep, so it's invertible

$\boxed{P_{\mathcal{B}} (\vec{x})_{\mathcal{B}} = \vec{x}}$ , so $\boxed{(\vec{x})_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x}}$

<u>Expl 6</u> $V = \mathbb{P}_2$

$p_1(t) = 1 + 2t^2$

$p_2(t) = 4 + t + 5t^2$

$p_3(t) = 3 + 2t$

Are $\{p_1, p_2, p_3\}$ lin. indep.?

$\mathcal{B} = \{1, t, t^2\}$, basis for $\mathbb{P}_2$

$$(p_1)_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad (p_2)_{\mathcal{B}} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \quad (p_3)_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{pmatrix} \xrightarrow[\text{reduction}]{\text{row}} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$(p_1)_{\mathcal{B}}, (p_2)_{\mathcal{B}}, (p_3)_{\mathcal{B}}$ are lin. dep.

$\Rightarrow p_1, p_2, p_3$ are $\boxed{\text{lin. dep.}}$ (change of coordinates is an isomorphism)

Check:

$$2(p_2)_{\mathcal{B}} - 5(p_1)_{\mathcal{B}} = (p_3)_{\mathcal{B}}$$

$$-5(p_1)_{\mathcal{B}} + 2(p_2)_{\mathcal{B}} - (p_3)_{\mathcal{B}} = \vec{0}$$

$$\Rightarrow \quad -5p_1 + 2p_2 - p_3 = 0$$

$$-5(1+2t^2) + 2(4+t+5t^2) - (3+2t)$$

$$= \cancel{-5} \cancel{-10t^2} \cancel{+8} \cancel{+2t} \cancel{+10t^2} \cancel{-3} \cancel{-2t} = 0 \checkmark$$