4.3 Linear independence and bases

Let V be a general vector space.

A set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is <u>linearly independent</u> if

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

"linear dependence relation"

cannot be rephrased as $A\vec{x}=\vec{0}$ unless $V=\mathbb{R}^n$

has only the trivial solution $c_1 = \cdots = c_p = 0$.

It is <u>linearly dependent</u> otherwise.

- $\{\vec{v}\}$ is lin. dep. iff $\vec{v}=\vec{0}$
- $\{\vec{v}_1, \vec{v}_2\}$ lin. dep. iff one is a multiple of the other
- $\{\vec{v}_1, \ldots, \vec{v}_p\}$ lin. dep. iff $\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$ for some j.

  $\neq \vec{0}$

<u>Expl 1</u>

$V = \mathbb{P}$ (space of polynomials)

$p_1(t) = 1, \quad p_2(t) = t, \quad p_3(t) = 4-t$

Are $\{p_1, p_2, p_3\}$ lin. dep. or indep.?

$$4p_1 - p_2 = 4(1) - t = 4 - t = p_3$$

$$4p_1 - p_2 - p_3 = 0$$

$\{p_1, p_2, p_3\}$ $\boxed{\text{lin. dep.}}$

Expl 2(a)

$V = C([0,1])$

$u_1 = \sin t, \quad u_2 = \cos t$

Are $\{u_1, u_2\}$ lin. dep. or lin. indep?

Suppose

$$c_1 u_1 + c_2 u_2 = 0$$

the zero function, so true for all $t \in [0,1]$

$$c_1 \sin t + c_2 \cos t = 0$$

Take $t=0$:

$$0 = c_1 \underbrace{\cancel{\sin 0}}_{0} + c_2 \underbrace{\cancel{\cos 0}}_{1} = c_2 \longrightarrow c_2 = 0$$

Take $t = \frac{\pi}{4}$:

$$0 = c_1 \underbrace{\cancel{\sin \tfrac{\pi}{4}}}_{\frac{\sqrt{2}}{2}} + \underbrace{\cancel{c_2}}_{0} \cos \frac{\pi}{4} = c_1 \cdot \frac{\sqrt{2}}{2} \longrightarrow c_1 = 0$$

So $c_1 = c_2 = 0$

$\boxed{\{u_1, u_2\} \text{ lin. indep.}}$

Expl 2(b)

$V = C([0,1])$

$v_1 = \sin t \cos t$, $v_2 = \sin 2t$

Are $\{v_1, v_2\}$ lin. dep. or indep.?

$$v_2 = \sin 2t \overset{\text{true for all } t \in [0,1]}{=} 2 \sin t \cos t = 2 v_1$$

So $\boxed{\{v_1, v_2\} \text{ lin. dep.}}$

<u>Defn</u> Let H be a subspace of V.
Let $\mathcal{B} \subset V$. Then $\mathcal{B}$ is a <u>basis for H</u> if

(i) $\mathcal{B}$ is lin. indep.

(ii) $\operatorname{Span} \mathcal{B} = H$.

- Common use is when $H = V$.
- If $\mathcal{B}$ is a basis for H, then $\mathcal{B} \subset H$ by (ii).

Expl 3

If A is $n \times n$ and invertible, then its columns are lin. indep. and span $\mathbb{R}^n$. So the columns form a basis for $\mathbb{R}^n$.

Expl 4

If $A = I_n$ in Expl 3, then its columns are

$$\underbrace{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n}_{\text{standard basis for } \mathbb{R}^n}$$

Expl 6

$$S = \overbrace{\{1, t, t^2, \ldots, t^n\}}^{\text{standard basis for } \mathbb{P}_n} \subset \mathbb{P}_n.$$

Verify that S is a basis for $\mathbb{P}_n$.

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lin indep?

$$c_0 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0$$ ← the zero poly.

Two polys are equal iff they have the same coeffs.

So $c_0 = c_1 = c_2 = \cdots = c_n = 0$ Yes, lin. indep. ✓

Span $\mathbb{P}_n$?

Let $p(t) \in \mathbb{P}_n$ be arbitrary. Then

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

for some a_k's in $\mathbb{R}$ (could be 0).

So $p(t) \in \operatorname{Span} S$. ✓

Thm 5 (Spanning set thm)

Let V be a vector space.

Let $S = \{\vec{v}_1, \dots, \vec{v}_p\} \subset V$.

Let $H = \operatorname{Span} S$.

(a) If some $\vec{v}_k$ is a lin. combo. of the other vectors in S, then

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_{k+1}, \dots, \vec{v}_p\} = H.$$

(b) If $H \neq \{\vec{0}\}$, then some subset of S is a basis for H.

Thm 6 The pivot columns of A form a basis for Col A. span of columns of A

Thm 7 If $A \sim B$, then Row A = Row B.

If B is in echelon form, the nonzero rows of B form a basis for Row A.

Expls 9 & 10

$$A = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{pmatrix}.$$

Find a basis for Col A.

Find a basis for Row A.

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row reduction

$$A \sim \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$$

pivot columns

$A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same soln set. That means the cols of A and the cols of B have the same lin. dep. relations

$$\text{basis for Col } A = \underbrace{\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 2 \\ 8 \end{pmatrix} \right\}}_{\text{pivot colums of } A}$$

$$\text{basis for Row } A = \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$