4.2 Null spaces, column spaces, row spaces, and linear transformations

$$\underbrace{\text{Nul } \overset{m\times n}{A}}_{\text{"null space of } A\text{"}} := \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

Expl 1

$$A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}. \quad \text{Is } \vec{u} \in \text{Nul } A?$$

$\vec{u} \in \text{Nul } A$ iff $A\vec{u} = \vec{0}$

$$A\vec{u} = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} \checkmark$$

$$\boxed{\text{Yes, } \vec{u} \in \text{Nul } A}$$

Thm

Nul A is a subspace of $\mathbb{R}^n$.

Pf:

- Nul A $\subset \mathbb{R}^n$ by defn. ✓

- $A\vec{0} = \vec{0}$, so $\vec{0} \in$ Nul A. ✓

- Suppose $\vec{u}, \vec{v} \in$ Nul A.
  Then $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$,
  so $\vec{u}+\vec{v} \in$ Nul A. ✓

- Suppose $\vec{u} \in$ Nul A and $c \in \mathbb{R}$.
  Then $A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$,
  so $c\vec{u} \in$ Nul A. $\square$

Expl 2

Let $H = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - 2b + 5c = d \text{ and } c - a = b \right\}$.

Show that H is a subspace of $\mathbb{R}^4$.

$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in H$ iff $\begin{aligned} a - 2b + 5c - d &= 0 \\ a + b - c \quad &= 0 \end{aligned}$

iff $\underbrace{\begin{pmatrix} 1 & -2 & 5 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $H = \text{Nul } A$, which makes H a subspace.

Expl 3

Find a spanning set for Nul A if

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

(This method is important.)

Nul A = soln set of $A\vec{x} = \vec{0}$

$$(A \mid \vec{0}) = \left(\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right)$$

row reduce to reduced echelon form

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\uparrow x_2$ $\uparrow x_4$ $\uparrow x_5$ free

$$x_1 - 2x_2 \quad - x_4 + 3x_5 = 0$$
$$x_3 + 2x_4 - 2x_5 = 0$$

Soln set:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathrm{Nul}\,A = \mathrm{Span} \underbrace{\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\boxed{\text{spanning set}}}$$

linear combo. of vect. in the spanning set. The weights are the free variables

- When using this method, the spanning set is lin. indep:

$$\left(\text{if } \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = \vec{0}, \text{ then } \overbrace{x_2 = x_3 = x_5}^{\text{weights}} = 0 \right)$$

- # of vectors in spanning set is # of free vars.

<u>Defn</u> If $A = (\vec{a}_1 \cdots \vec{a}_n)$ is $m \times n$, then

$\underbrace{\text{Col} A}_{\text{"column space of } A\text{"}} = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} \subset \mathbb{R}^m.$

<u>Thm 3</u> Col A is a subspace of $\mathbb{R}^m$.

- If $T(\vec{x}) = A\vec{x}$, then Col A = range of T.

<u>Expl 4</u>

$$W = \left\{ \begin{pmatrix} 6a-b \\ a+b \\ -7a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Find an A so that $W = \text{Col} A$.

$$W = \left\{ a\begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{A = \begin{pmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{pmatrix}}$$

Defn If $A = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{pmatrix}$ is $m \times n$, then

$\underbrace{\text{Row } A}_{\text{"row space of } A\text{"}} = \text{Span}\{\vec{r}_1, \dots, \vec{r}_m\} \subset \mathbb{R}^n$

- $\text{Row } A = \text{Col } A^T$

[Look at table on p. 217]

Defn Let V, W be vector spaces. A function $T: V \to W$ is linear (or a linear transformation) if

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall \vec{u}, \vec{v} \in V,$

(ii) $T(c\vec{u}) = c\,T(\vec{u}) \quad \forall \vec{u} \in V,\ c \in \mathbb{R}$

- The range of T is $\{T(\vec{u}) : \vec{u} \in V\} \subset W$, as usual. If T is linear, the range of T is a subsp. of W. If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then $T(\vec{x}) = A\vec{x}$ for some A, and range of $T = \text{Col } A$.

- If T is linear, the <u>kernel</u> (or <u>null space</u>) of T is $\{\vec{u} \in V : T(\vec{u}) = \vec{0}\} \subset V$.

The kernel is a subspace of V.

If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then $T(\vec{x}) = A\vec{x}$ for some A, and kernel of T = Nul A.

<u>Expl</u>

$V = C^k([a,b])$
$= \{ f : [a,b] \to \mathbb{R} \mid \frac{d^k f}{dx^k} \text{ exists and is continuous} \}$

(If $k=0$, write $C([a,b])$ and use $\frac{d^0 f}{dx^0} = f$.)

V is a vector space.

<u>Expl 9</u>

$V = C^1([a,b])$, $W = C([a,b])$

Define $D : V \to W$ by $D(f) = f'$.

Show that D is linear.

What is the kernel and range of D?

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ ✓

$D(cf) = (cf)' = cf' = cD(f)$ ✓

So $\boxed{D \text{ is linear}}$

$D(f) = 0$ iff $f' = 0$ iff f is constant

$$\boxed{\text{kernel of } D = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is constant} \}}$$

Given $f \in W = C([a,b])$, there is an $F \in V = C^1([a,b])$ s.t.

$D(F) = F' = f$. (e.g. $F(x) = \int_a^x f(t)\,dt$)

So D is onto. $\boxed{\text{range of } D = W}$

<u>Expl 10</u>

$V = C^2([a,b])$, $W = C([a,b])$

$\omega \in \mathbb{R}$ is a constant

Define $T: V \to W$ by $T(f) = f'' + \omega^2 f$.

Show that T is linear.

Find the kernel of T.

$$T(f+g) = (f+g)'' + \omega^2 (f+g)$$
$$= f'' + \omega^2 f + g'' + \omega^2 g$$
$$= T(f) + T(g) \checkmark$$

$$T(cf) = (cf)'' + \omega^2 (cf)$$
$$= cf'' + \omega^2 cf = c(f'' + \omega^2 f)$$
$$= c\,T(f) \checkmark$$

So $\boxed{T \text{ is linear}}$

$T(f) = 0$ iff $f'' + \omega^2 f = 0$

needs diff. eqns. $\rightarrow$ iff $f(x) = c_1 \sin(\omega t) + c_2 \cos(\omega t)$ for some $c_1, c_2 \in \mathbb{R}$

$v_1(t) = \sin(\omega t)$, $v_2(t) = \cos(\omega t)$

$v_1, v_2 \in V = C^2([a,b])$

$T(f) = 0$ iff $f = c_1 v_1 + c_2 v_2$ for some c_1, c_2

So $\boxed{\text{kernel of } T = \operatorname{Span}\{v_1, v_2\}}$