4.1 Vector spaces and subspaces

A <u>(real) vector space</u> is a nonempty set V, together with two operations, addition and scalar multiplication, satisfying the following 10 axioms:

(1) if $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$

(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(4) $\exists \vec{0} \in V$ s.t. $\vec{u} + \vec{0} = \vec{u} \quad \forall \vec{u}$

(5) $\forall \vec{u} \in V$, $\exists -\vec{u} \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$

(6) if $\vec{u} \in V$ and $c \in \mathbb{R}$, then $c\vec{u} \in V$

(7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(8) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

(9) $c(d\vec{u}) = (cd)\vec{u}$

(10) $1\vec{u} = \vec{u}$

- Can prove that $\vec{0}$ and $-\vec{u}$ are unique.
- Can prove that $0\vec{u} = \vec{0}$, $c\vec{0} = \vec{0}$, $(-1)\vec{u} = -\vec{u}$

- A "complex vector space" is the same, but scalars are in $\mathbb{C}$. We will assume scalars are in $\mathbb{R}$, but all theorems in Ch. 4 are also true for complex vector spaces.

<u>Expl 1</u>

$\mathbb{R}^n$ is a vector space.

<u>Expl 5</u>

Let A be a set.
Let V be the set of functions $f: A \to \mathbb{R}$.

For $f, g \in V$, define $f+g \in V$ by

$$(f+g)(x) = f(x) + g(x).$$

For $f \in V$ and $c \in \mathbb{R}$, define $cf \in V$ by

$$(cf)(x) = cf(x).$$

Then V is a vector space.
(The zero vector here is the zero function.)

Expl 3

In the last example, let $A = \mathbb{Z}$.

If $y \in V$, then $y : \mathbb{Z} \to \mathbb{R}$.

Write y_k instead of $y(k)$.

Then y is a "doubly infinite sequence":

$$\{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$$

Expl 4

Let $n \geq 0$.

$\mathbb{P}_n$ = set of all polynomials of degree $\leq n$.

(also includes the zero polynomial, whose degree is undefined in our book)

$\mathbb{P}_n$ is a vector space

A <u>subspace</u> of a vector space V is a subset H of V s.t.

(a) $\vec{0} \in H$

(b) if $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$

(c) if $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$

- Every subspace is a vector space.
- V is a subspace of itself.

Expl 6

If V is a vector sp., then $\{\vec{0}\}$ is a subspace.

It's called the zero subspace.

Expl 7

$\mathbb{P}$ = set of all polynomials (of any degree)

$\mathbb{P}$ is a vector space.

Is $\mathbb{P}_n$ a subspace of $\mathbb{P}$?

$\mathbb{P}_n \subset \mathbb{P}$ ✓

zero polynomial is in $\mathbb{P}_n$ ✓

$\mathbb{P}_n$ is closed under addition and scalar multiplication ✓

$\boxed{\text{Yes, } \mathbb{P}_n \text{ is a subspace of } \mathbb{P}.}$

Expl 9

$$H = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

(a) Is H a subspace of $\mathbb{R}^3$?

(b) Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

(a) $H \subset \mathbb{R}^3$ ✓

$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in H$ ✓

H is closed under operations ✓

$\boxed{\text{Yes, } H \text{ is a subspace of } \mathbb{R}^3}$

(b) $\boxed{\text{No, } \mathbb{R}^2 \text{ is not a subset of } \mathbb{R}^3}$

linear combination and span can be defined as usual in a general vector sp.

Expl

Let V be a vector sp. Let $\vec{v}_1, \vec{v}_2 \in V$.
Show that $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ is a subspace of V.

$H = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$

$H \subset V$ ✓

$\vec{0} \in H$?

$\vec{0} = 0\,\vec{v}_1 + 0\,\vec{v}_2 \in H$ ✓

Closed under addition?

$\vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$?

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$\vec{v} = d_1 \vec{v}_1 + d_2 \vec{v}_2$$

$$\vec{u} + \vec{v} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 \in H \checkmark$$

Closed under scalar multiplication?

$\vec{u} \in H$, $c \in \mathbb{R}$

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$c\vec{u} = (cc_1)\vec{v}_1 + (cc_2)\vec{v}_2 \in H \checkmark$$

Thm 1 If $\vec{v}_1, \dots, \vec{v}_p \in V$, then $\underbrace{\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}}_{\text{"subspace spanned (or generated) by } \vec{v}_1, \dots, \vec{v}_p\text{"}}$ is a subspace of V.

Let $H \subset V$ be a subspace and $\{\vec{v}_1, \dots, \vec{v}_p\} \subset V$.
If $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\} = H$, then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a spanning (or generating) set for H.

Expl 12

$$H = \left\{ \begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$ Show that H is a subspace of $\mathbb{R}^4$.

$$\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$H = \left\{ a \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$

By Thm 1, H is a subspace.