# 3.2 Properties of determinants

$A$ : a square matrix

$$\left.\begin{array}{lll}
A \xrightarrow[R_i + cR_j \to R_i]{} B : & & \det B = \det A \\
 & & \\
A \xrightarrow[R_i \leftrightarrow R_j]{} B : & & \det B = -\det A \\
 & & \\
A \xrightarrow[cR_i \to R_i,\ c \neq 0]{} B : & & \det B = c \det A \\
 & & \left(\det A = \frac{1}{c} \det B\right)
\end{array}\right\} \text{Thm 3}$$

Expl 1

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}. \quad \text{Find } \det A.$$

$$\det A \underset{R_2 + 2R_1 \to R_1}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} \underset{R_2 + R_1 \to R_3}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{=} - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} \underset{\text{triangular}}{=} -(1)(3)(-5) = \boxed{15}$$

Expl 2

$$A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}. \quad \text{Find } \det A.$$

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$$\det A \underset{\frac{1}{2}R_1 \to R_1}{=} \overset{\frac{1}{c}}{2} \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$c = \frac{1}{2}$

$$\underset{\substack{R_2 - 3R_1 \to R_2 \\ R_3 + 3R_1 \to R_3 \\ R_4 - R_1 \to R_4}}{=} 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$\underset{R_3 + 4R_2 \to R_3}{=} 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} \underset{R_4 - \frac{1}{2}R_3 \to R_4}{=} 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{\text{triangular}}{=} 2(1)(3)(-6)(1) = \boxed{-36}$$

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$$n\times n \quad A \xrightarrow[\text{(always possible)}]{\text{row reduction using only row replacement and interchange}} U \quad n\times n$$

echelon form

$$\det A = (-1)^r \det U$$

$r$: # of interchanges

$A$ invertible : $U$ is upper triangular (pivot in every row/col)

$\det U = \underbrace{\text{product of pivots}}_{\text{nonzero}}$

$\therefore \det A \neq 0$

$A$ singular : $U$ has a zero row

$\det U = 0$ (exp. across last row)

$\therefore \det A = 0$

$$\boxed{\begin{array}{l}\underline{\text{Thm 4}} \text{ If } A \text{ is a square matrix, then} \\ A \text{ is invertible iff } \det A \neq 0.\end{array}}$$

$\det A = 0$ iff $A$ is singular

iff cols of $A$ lin. dep.

iff $A^T$ is singular

iff cols of $A^T$ lin. dep.

iff rows of $A$ lin. dep.

## Expl 3

$$A = \begin{pmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{pmatrix}.$$ Find det A.

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$$\det A \underset{R_3+2R_1\to R_3}{=} \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} \underset{\substack{\text{rows are lin. dep.} \\ (R_2 = R_3)}}{=} \boxed{0}$$

## Expl 4

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}.$$ Find det A.

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Soln 1

$$\det A \underset{R_4+R_2\to R_4}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

$$\underset{\substack{\text{exp. across} \\ 1^{st}\text{ col}}}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} \underset{R_2-3R_1\to R_2}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2(-5)\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = 10(-3-0) = \boxed{-30}$$

exp. across 2nd row

Soln 2

$$\det A = \cdots = -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

starts the same

$$= 2\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 2(1)(-3)(5) = \boxed{-30}$$

$R_2 \leftrightarrow R_3$ triangular

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Thm 5 If $A$ is an $n\times n$ matrix, then $\det A^T = \det A$.

Pf: If $n=1$, then $A^T = A$, so $\det A^T = \det A$. Suppose the theorem is true for $n=k$ and let $n=k+1$. By defn,

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}, \quad (1)$$

where $C_{ij} = (-1)^{i+j}\det A_{ij}$. Let $A^T = (\tilde{a}_{ij})$ with cofactors $\tilde{C}_{ij}$. By a cofactor expansion along the 1st col,

$$\det A^T = \tilde{a}_{11}\tilde{C}_{11} + \tilde{a}_{21}\tilde{C}_{21} + \cdots + \tilde{a}_{n1}\tilde{C}_{n1}.$$

But $\tilde{a}_{ij} = a_{ji}$, so

$$\det A^T = a_{11}\tilde{C}_{11} + a_{12}\tilde{C}_{21} + \cdots + a_{1n}\tilde{C}_{n1}. \quad (2)$$

Also, $\tilde{C}_{ij} = (-1)^{i+j}\det(A^T)_{ij}$. Since $(A^T)_{ij} = (A_{ji})^T$, and $A_{ji}$ is $k\times k$, and the theorem is true for $n=k$, we have $\det(A_{ji})^T = \det A_{ji}$. Thus, $\tilde{C}_{ij} = (-1)^{i+j}\det A_{ji} = (-1)^{j+i}\det A_{ji} = C_{ji}$. Therefore, (2) becomes

$$\det A^T = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

By (1), we get $\det A = \det A^T$. By induction, the theorem is true for all $n$. $\square$

## Thm 6

If $A$ and $B$ are $n \times n$, then $\det(AB) = (\det A)(\det B)$.

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**Warning:** In general, $\det(A+B) \neq \det A + \det B$

## Expl 5

$A = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

Verify Thm 6 for $A$ and $B$.

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$\det A = 12 - 3 = 9$, $\det B = 8 - 3 = 5$

$AB = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 25 & 20 \\ 14 & 13 \end{pmatrix}$

$\det(AB) = 25(13) - 20(14) = 5(5 \cdot 13 - 4 \cdot 14)$

$= 5(65 - 56) = 5 \cdot 9 = 45 = (\det A)(\det B)$ ✓