3.1 Introduction to determinants

A : a square matrix

A_{ij} : the submatrix obtained by deleting the i^{th} row and j^{th} column

Expl

$$A = \begin{pmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{pmatrix} \qquad A_{32} = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

The <u>(i,j)-cofactor</u> of A is

$$c_{ij} = (-1)^{i+j} \det A_{ij}$$

"determinant" (about to be defined)

$(-1)^{i+j}$:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The <u>determinant</u> of an $n \times n$ matrix, A, written $\det A$, is defined recursively:

for 1×1: $A = (a_{11})$, $\det A = a_{11}$

for $n \times n$:

involve determinants of $(n-1) \times (n-1)$

$$\det A = \underbrace{a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}}_{\text{cofactor expansion across the first row}}$$

<u>Thm 1</u>

For any i,

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

cofactor exp. across i^{th} row

For any j,

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

cofactor exp. across j^{th} col

<u>Notation</u>:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

* This definition gives us $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Expls 1 & 2

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}. \quad \text{Find } \det A.$$

Soln 1

Cofactor exp across 1^{st} row:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det A = +1 \det\begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} - 5 \det\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$$

$$+ 0 \det\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}$$

$$= (0-2) - 5(0-0) + 0(-4-0) = \boxed{-2}$$

Soln 2

Cofactor exp. across 2^{nd} col.

$$\det A = -5 \det\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} + 4 \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$-(-2) \det\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$= -5(0-0) + 4(0-0) + 2(-1-0) = \boxed{-2}$$

Soln 3:

Cofactor exp. across 3rd row:

$$\det A = +0\cancel{\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\cancel{\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}$$

$$= 2(-1-0) = \boxed{-2}$$

Expl 3

$$A = \begin{pmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}.$$ Find det A.

$$\det A \overset{\text{along 1st col.}}{=} 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

$$\overset{\text{along 1st col.}}{=} 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$\overset{\text{along } 3^{rd} \text{ row}}{=} 3 \cdot 2 \cdot \left(-(-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \right)$$

$$= 12(-1-0) = \boxed{-12}$$

<u>Thm 2</u> If A is a triangular matrix, then $\det A$ is the product of the diagonal entries.

Upper triangular: $A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}$

Lower triangular: $A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ * & & \ddots \\ & & & a_{nn} \end{pmatrix}$

a_{kk} and $*$ can be anything

In either case, $\det A = a_{11} a_{22} \cdots a_{nn}$.