2.3 Characterization of invertible matrices

Thm 8 (The invertible matrix theorem)

Let A be an $n \times n$ matrix. Then TFAE:

(a) A is invertible

(b) A is row equiv. to I

(c) A has n pivot positions

(d) $A\vec{x} = \vec{0}$ has only the trivial soln

(e) The cols of A are lin. indep.

(f) $T(\vec{x}) = A\vec{x}$ is one-to-one

(g) $A\vec{x} = \vec{b}$ is consistent $\forall \vec{b} \in \mathbb{R}^n$

(h) The cols of A span $\mathbb{R}^n$

(i) $T(\vec{x}) = A\vec{x}$ is onto

(j) There is an $n \times n$ matrix C with $CA = I$

(k) There is an $n \times n$ matrix D with $AD = I$

(l) A^T is invertible

Thm Let A, B be square. If $AB = I$, then A and B are invertible, $A^{-1} = B$, and $B^{-1} = A$.

Pf: Assume $AB = I$. By Thm 8(k), A is invertible, and $A^{-1} = A^{-1}I = A^{-1}AB = IB = B$. By Thm 8(j), B is invertible and $B^{-1} = IB^{-1} = ABB^{-1} = AI = A$. $\square$

Expl 1

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}.\quad \text{Is } A \text{ invertible?}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \xrightarrow[R_3 + 5R_1 \to R_3]{R_2 - 3R_1 \to R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 + R_2 \to R_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

A has 3 pivot positions (and A is 3×3), so $\boxed{A \text{ is invertible}}$ by Thm 8(c).

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto, then T is invertible and has an inverse $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$T(T^{-1}(\vec{x})) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$
$$T^{-1}(T(\vec{x})) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$

Thm 9

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be linear w/ matrix A. Then T is invertible iff A is invertible. If T is invertible, then T^{-1} is also linear and has matrix A^{-1}.

Pf: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be linear. Assume T is invertible. Then T is one-to-one and onto, so by Thm 8(f) (or Thm 8(i)), A is invertible.

Now assume A is invertible. By Thm 8(f) and 8(i), T is one-to-one and onto, so T is invertible. Let $\vec{x} \in \mathbb{R}^n$ and $\vec{y} = T^{-1}(\vec{x})$. Then $T(\vec{y}) = \vec{x}$, which means $A\vec{y} = \vec{x}$.

Thus,
$$T^{-1}(\vec{x}) = \vec{y} = I\vec{y} = A^{-1}A\vec{y} = A^{-1}\vec{x},$$
so T^{-1} is linear, and has matrix A^{-1}. □

Expl 2

What can you say about $T: \mathbb{R}^n \to \mathbb{R}^n$ if you know it's linear and one-to-one.

Since it's linear, $T(\vec{x}) = A\vec{x}$ for some $n \times n$ matrix A. By Thm 8(f), A is invertible. So by Thm 9, T is invertible. In other words $\boxed{T \text{ is both one-to-one } \underline{\text{and}} \text{ onto.}}$

In a similar way, using Thm 8(i),

$$\boxed{\text{if } T: \mathbb{R}^n \to \mathbb{R}^n \text{ is linear and onto, then it's also one-to-one}}$$