2.2 The inverse of a matrix

Let A be $n \times n$. If C is $n \times n$ and $CA = AC = I$, then C is called an <u>inverse</u> of A.

<u>Thm</u> If A has an inverse, then the inverse is unique.

<u>Pf:</u> Suppose B and C are both inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C. \square$$

By the thm, we can speak of <u>the</u> inverse of A (if it exists). We denote it by A^{-1}.

- If A^{-1} exists, then A is <u>invertible</u> or <u>nonsingular</u>.
- If A^{-1} does not exist, then A is <u>singular</u>.

<u>Expl 1</u>

$A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$. Verify that $A^{-1} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$.

$$\left.\begin{array}{l} \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark \\ \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark \end{array}\right\}$$

must check both, according to the defn.

(After Thm. 8 in Sect. 2.3, we will only have to check one of these)

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2×2, then the <u>determinant</u> of A is

$$\det A = ad - bc$$

(We will study determinants in Ch. 3)

<u>Thm 4</u> If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A is invertible iff $\det A \neq 0$, in which case

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

---

memory hint: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

<u>Expl</u> $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$. Find A^{-1}!

$$\det A = 3(6) - 4(5) = 18 - 20 = -2$$

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix}}$$

<u>Thm 5</u> If A is invertible, then $A\vec{x} = \vec{b}$ has one soln, and it is $\vec{x} = A^{-1}\vec{b}$.

<u>Pf:</u> Suppose A is invertible and define $\vec{x} = A^{-1}\vec{b}$. Then $A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b}$

$= I\vec{b} = \vec{b}$, so $\vec{x}$ is a soln.

Suppose $\vec{u}$ is another soln, so that $A\vec{u} = \vec{b}$.

Then

$$\vec{u} = I\vec{u} = (A^{-1}A)\vec{u} = A^{-1}(A\vec{u}) = A^{-1}\vec{b} = \vec{x},$$

so the soln is unique. $\square$

<u>Expl 4</u>

Solve the system

$$3x_1 + 4x_2 = 3$$
$$5x_1 + 6x_2 = 7$$

from Expl 2

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Want to solve $A\vec{x} = \vec{b}$

$\det A = 18 - 20 = -2$, so A is invertible

$$A^{-1} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix}$$

System has the unique soln

$$A^{-1}\vec{b} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ -3 \end{pmatrix}}$$

$$\frac{15}{2} - \frac{21}{2} = -\frac{6}{2}$$

Thm 6

(a) If A is invertible, then so is A^{-1}, and

$$(A^{-1})^{-1} = A.$$

(b) If A and B are invertible, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(c) If A is invertible, then so is A^T, and

$$(A^T)^{-1} = (A^{-1})^T.$$

Pf: (a) Need to find C so that

$$A^{-1}C = CA^{-1} = I.$$

Since A is invertible, this is true for $C = A$.

Thus, A^{-1} is invertible, and its inverse is A.

(b) Let $C = B^{-1}A^{-1}$. Need to show that

$$(AB)C = C(AB) = I.$$

First,

$$(AB)C = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Next,

$$C(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

(c) Let $C = (A^{-1})^T$. Need to check that

$$A^TC = CA^T = I.$$

First,

$$A^TC = A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Next,

$$CA^T = (A^{-1})^TA^T = (AA^{-1})^T = I^T = I. \quad \square$$

An <u>elementary matrix</u> is a matrix obtained from I by using a single row operation

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - 4R_1 \to R_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}}_{\text{an elementary matrix}}$

other examples: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$$\begin{tikzcd} & E \\ I \arrow[ur, bend left, "\text{row operation}"] \arrow[dr, bend right, "\text{reverse row operation}"'] & \\ & E^{-1} \end{tikzcd}$$

elementary matrices are invertible

$$\begin{tikzcd} I \arrow[r, "\text{row oper.}"] & E \arrow[d] \\ A \arrow[r, "\text{some row oper.}"] & A' \end{tikzcd}$$

$A' = EA$

row operations can be done by left multiplying by an elementary matrix.

<u>Thm 7</u> A is invertible iff A is row equivalent to I. If A is invertible, then any sequence of row operations that transforms A into I will transform I into A^{-1}.

<u>Pf:</u> Suppose A is invertible. Let A' be its reduced echelon matrix. Since $A\vec{x} = \vec{b}$ has

a soln for all $\vec{b}$, A' has a pivot in every row. But A is square, so $A' = I$. Thus, A is row equivalent to I.

Now suppose A is row equivalent to I. Then there is a sequence, $E_1, E_2, \ldots, E_p$, of elem. matrices such that

$$E_p \cdots E_2 E_1 A = I.$$

Let $B = E_p \cdots E_1$. Since each E_j is invertible, B is invertible. (Thm 6b) Thus,

$$A = IA = B^{-1}BA = B^{-1}I = B^{-1}.$$

By Thm 6a, B^{-1} is invertible, so A is invertible. If we apply the same row operations to I, we get

$$BI = B = (B^{-1})^{-1} = A^{-1}. \quad \square$$

<u>Algorithm for finding A^{-1}</u>

Reduce $(A \mid I)$ to reduced echelon form. If you get something that looks like $(I \mid B)$, then A is invertible, and $B = A^{-1}$. Otherwise, A is singular.

Expl 7

$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$. Find A^{-1}, if it exists.

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \leftrightarrow R_2$

$R_3 - 4R_1 \to R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right)$$

$R_3 + 3R_2 \to R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right)$$

$\frac{1}{2}R_3 \to R_3$

$R_2 - 2R_3 \to R_2$

$R_1 - 3R_3 \to R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}}$$