Test 1 Review

Sections covered: 1.1 – 1.5, 1.7 – 1.9, 2.1 – 2.3

Systems of lin. eqns.

$$\left.\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

Possibilities:

- no solution (inconsistent)
- one solution
- infinitely many solutions

(one solution / infinitely many solutions: consistent)

augmented matrix

$$\overbrace{\left(\begin{array}{ccc|c} \underbrace{\begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}}_{\text{coefficient matrix}} & & & \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \end{array}\right)}^{\text{augmented matrix}}$$

bar is optional

coefficient matrix

Row operations:

- Replacement ($R_i + cR_j \to R_i$)
- Interchange ($R_i \leftrightarrow R_j$)
- Scaling ($cR_i \to R_i$, $c \neq 0$)

(all are reversible)

$A \sim A'$ "row equivalent"

if A can be turned into A' using row operations

If $(A \mid \vec{b}) \sim (A' \mid \vec{b}')$, then both systems have the same solution set.

Concepts to review:

- echelon form
- reduced echelon form (unique)
- pivot positions
- pivot columns
- row-reduction algorithm
- free variables

$$(A \mid \vec{b}) \sim \begin{pmatrix} & & & * & \\ 0 & 0 & \cdots & 0 & \boxed{\neq 0} \\ & & & * & \end{pmatrix}$$

means system is inconsistent; otherwise, consistent

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ← a vector

$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ ← a lin. combo. of $\vec{v}_1, \ldots, \vec{v}_p$

$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ = set of all lin. combos of $\vec{v}_1, \ldots, \vec{v}_p$

$$A\vec{x} = (\vec{a}_1 \cdots \vec{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

need $\underset{m \times n}{A}$ $\underset{n \times 1}{\vec{x}}$

- review algorithm for calculating

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lin. system:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

augmented matrix $\rightarrow (\vec{a}_1 \cdots \vec{a}_n \mid \vec{b})$

$\updownarrow$

vector eqn: $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$

$\updownarrow$

matrix eqn.: $A\vec{x} = \vec{b}$

all 3 have same soln set

Cols of A span $\mathbb{R}^m$

$\Updownarrow$

$A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$

$\Updownarrow$

A has a pivot in every row

$A\vec{x} = \vec{0} \leftarrow$ homogen

$\vec{x} = \vec{0}$ is the trivial soln

Nontrivial solns exist iff there are free vars.

$\vec{v}_1, \ldots, \vec{v}_p$ are lin. indep.

iff

$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ has only the trivial soln, $c_1 = \cdots = c_p = 0$

Cols of A lin. indep.

iff

$A\vec{x} = \vec{0}$ has only the trivial soln

$\{\vec{v}\}$ lin. dep. $\Longleftrightarrow$ $\vec{v} = \vec{0}$

$\{\vec{v}_1, \vec{v}_2\}$ lin. dep. $\Longleftrightarrow$ one is a multiple of the other

$\{\vec{v}_1, \ldots, \vec{v}_p\}$ lin. dep $\Longleftrightarrow$ $\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$ for some j

$\neq \vec{0}$

$S = \{\vec{v}_1, \dots, \vec{v}_p\} \subset \mathbb{R}^n$

If $p > n$, then S is lin. dep.

OR

If A is $n \times p$ and $p > n$,

then cols of A are lin. dep.

If $\vec{0} \in S$, then S is lin. dep.

$T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\vec{u}) = cT(\vec{u})$

If T is linear, then $T(\vec{x}) = A\vec{x}$, where

$$\underbrace{A}_{m \times n} = \left(T(\vec{e}_1) \ \cdots \ T(\vec{e}_n) \right)$$

T is one-to-one $\iff \begin{cases} A\vec{x} = \vec{0} \text{ has only the trivial soln} \\ \text{OR} \\ \text{The cols of } A \text{ are lin. indep.} \end{cases}$

range of T = span of cols of A

If $n = m$, then

T is one-to-one $\Longleftrightarrow$ T is onto

If $m > n$ then T is not onto

(and cols of A don't span $\mathbb{R}^m$)

$$A = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{pmatrix}, \quad B = (\vec{b}_1 \cdots \vec{b}_p)$$

$$AB = (A\vec{b}_1 \cdots A\vec{b}_p) = \begin{pmatrix} \vec{r}_1^{\,T} B \\ \vdots \\ \vec{r}_m^{\,T} B \end{pmatrix}$$

review algorithm for computing AB

$$\left.\begin{array}{l} T \text{ has matrix } A \\ S \quad " \quad\quad " \quad B \end{array}\right\} \Rightarrow \left\{\begin{array}{l} T \circ S \text{ has matrix } AB \\ T^{-1} \quad " \quad\quad " \quad A^{-1} \end{array}\right.$$

$$(AB)^T = B^T A^T, \quad (AB)^{-1} = B^{-1} A^{-1}$$

$$(A^T)^T = A, \quad (A^{-1})^{-1} = A, \quad (A^T)^{-1} = (A^{-1})^T$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = ad - bc,$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Review:

- elementary matrices
- algorithm for finding A^{-1}
- the invertible matrix theorem

Review it in groups: (a)-(c),
(d)-(f),
(g)-(i),
(j)-(k),
(l)