2.1 Matrix operations

Notation and terminology

- a_{ij} is the entry in row i, column j of A

  or A_{ij}

- we write $A = (a_{ij})$
- diagonal entries are $a_{11}, a_{22}, a_{33}, \dots$
- they form the main diagonal of A
- A diagonal matrix is a square matrix ($n \times n$) where every nondiagonal entry is 0.
- A zero matrix is a matrix of all 0's. It is written simply as "0", with its dimensions determined by context.

Matrix addition

If $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$,

then $A + B = (a_{ij} + b_{ij})$

Scalar multiplication

If $A = (a_{ij})$ and r is a scalar,

then $rA = (ra_{ij})$

$-A := (-1)A$

$A - B := A + (-B)$

Vectors are matrices (w/ one column)

These defns are consistent w/ the defns for vectors

Expl 1

$A = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$

Find $A+B$ and $A+C$, if they are defined.

$$A + B = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{pmatrix}}$$

A, C not same dimensions (or size), so $\boxed{A + C \text{ is undefined}}$

Expl 2

With A, B from Expl 1, find $A - 2B$.

$$A - 2B = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{pmatrix}}$$

Thm 1

Let A, B, C all have the same size and let r, s be scalars.

(a) $A + B = B + A$

(b) $(A+B) + C = A + (B + C)$

(c) $A + 0 = A$

(d) $r(A+B) = rA + rB$

(e) $(r+s)A = rA + sA$

(f) $r(sA) = (rs)A$

Matrix multiplication

Def: If A is $m \times n$ and $B = (\underset{\text{in } \mathbb{R}^n}{\vec{b}_1} \cdots \vec{b}_p)$ is $n \times p$, then AB is the $m \times p$ matrix defined by

$$AB = A(\vec{b}_1 \cdots \vec{b}_p) := (A\vec{b}_1 \; \underbrace{A\vec{b}_2}_{\text{in } \mathbb{R}^m} \cdots A\vec{b}_p)$$

Expl 3

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}.$$

Find AB, if it is defined.

A is 2×2, B is 2×3

match, so AB is defined

dimensions of AB (2×3)

$$B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$
$$\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$$

$$AB = A(\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3) = (A\vec{b}_1 \ A\vec{b}_2 \ A\vec{b}_3)$$

$$A\vec{b}_1 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$$

$$A\vec{b}_2 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$$

$$A\vec{b}_3 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ -9 \end{pmatrix}$$

$$\boxed{AB = \begin{pmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{pmatrix}}$$

Expl A is 3×5, B is 5×2.
What are the sizes of AB and BA, if they are defined?

$$\begin{array}{cc} A & B \\ 3\times 5 & 5\times 2 \end{array}$$

defined

$$\boxed{AB \text{ is } 3\times 2}$$

$$\begin{array}{cc} B & A \\ 5\times 2 & 3\times 5 \end{array}$$

$$\boxed{BA \text{ undefined}}$$

Thm (row-column rule for computing AB)
Let $A=(a_{ij})$ be $m\times n$ and $B=(b_{ij})$ be $n\times p$.
If $C=AB$, then $C=(c_{ij})$, where

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$$

$$= a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}.$$

Pf: Write $B=(\vec{b}_1 \cdots \vec{b}_p)$, where

$$\vec{b}_j=\begin{pmatrix} b_{1j}\\ b_{2j}\\ \vdots \\ b_{nj}\end{pmatrix}.$$ Then $C=AB=(A\vec{b}_1 \cdots A\vec{b}_p)$.

Thus, c_{ij} is the ith entry of $A\vec{b}_j$. By the row-vector rule for computing $A\vec{x}$, we get

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}. \quad \square$$

<u>Expl 5</u>

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$ from Expl 3

Let $C = AB$. Find c_{13} and c_{22}.

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c_{13}

$$C = AB = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

$$c_{13} = 2(6) + 3(3) = 12 + 9 = \boxed{21}$$

$$C = AB = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

c_{22}

$$c_{22} = 1(3) + (-5)(-2) = 3 + 10 = \boxed{13}$$

<u>Thm</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^p \to \mathbb{R}^n$ be linear transformations with matrices A and B, respectively. (So A is $m \times n$ and B is $n \times p$.) Then

$T \circ S : \mathbb{R}^p \to \mathbb{R}^m$ is a linear transformation whose matrix is AB.

Pf: Let $R = T \circ S : \mathbb{R}^p \to \mathbb{R}^m$.

$$R(\vec{u}+\vec{v}) = T(S(\vec{u}+\vec{v})) = T(S(\vec{u}) + S(\vec{v}))$$
$$= T(S(\vec{u})) + T(S(\vec{v})) = R(\vec{u}) + R(\vec{v}) \checkmark$$

$$R(c\vec{u}) = T(S(c\vec{u})) = T(c\,S(\vec{u}))$$
$$= c\,T(S(\vec{u})) = c\,R(\vec{u}) \checkmark$$

So R is linear. ✓

The matrix of R is an $m \times p$ matrix, and can be written as $(R(\vec{e}_1) \cdots R(\vec{e}_p))$. Now

$$S(\vec{e}_j) = B\vec{e}_j = (\vec{b}_1 \cdots \vec{b}_p)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j^{\text{th}} \text{ spot} \\ \\ \\ \\ \end{array}$$

$$= \vec{b}_j,$$

so that

$$R(\vec{e}_j) = (T \circ S)(\vec{e}_j) = T(S(\vec{e}_j)) = T(\vec{b}_j) = A\vec{b}_j.$$

Thus, the matrix of R is

$$(A\vec{b}_1 \quad A\vec{b}_2 \quad \cdots \quad A\vec{b}_p) = AB,$$

according to the definition of matrix multiplication. $\square$

In terms of matrices alone, this thm says:

$$\boxed{A(B\vec{x}) = (AB)\vec{x}}$$

Thm 2

$$\left.\begin{array}{ll} \text{(a)} & A(BC) = (AB)C \\ \text{(b)} & A(B+C) = AB + AC \\ \text{(c)} & (B+C)A = BA + CA \\ \text{(d)} & r(AB) = (rA)B = A(rB) \end{array}\right\} \text{provided the operations are defined}$$

scalar r

(e) If A is $m \times n$, then $I_m A = A I_n = A$

Expl 7

$$A = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$$

Show that $AB \neq BA$.

$$AB = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 3 \\ -2 & -6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 29 & -2 \end{pmatrix}$$

not equal

Warnings:

In general ...

- $AB \neq BA$
- $AB = AC \not\Rightarrow B = C$
- $AB = O \not\Rightarrow A = O$ or $B = O$

If $AB = BA$, we say A and B commute

If A is $n \times n$, then $A^k = \underbrace{AA \cdots A}_{k \text{ factors}}$

and $A^0 = I_n$

If $\underset{m \times n}{A} = (a_{ij})$, then $\underset{n \times m}{A^T} = (a_{ji})$

A^T is called the transpose of A.

Expl 8

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{pmatrix}$$

Find A^T, B^T, C^T.

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad B^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{pmatrix}$$

<u>Thm 3</u>

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

(d) $(AB)^T = B^T A^T$

<u>Pf of (d):</u> $A : m\times n$, $B : n\times p$

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

$$(B^TA^T)_{ij} = \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^{n} B_{ki} A_{jk}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki} \,. \quad \square$$

If $\vec{x}$ is a (column) vector, then $\vec{x}^T$ is a <u>row vector</u>.

Expl

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix} \leftarrow \text{from Expl 3}$$

Find the second row of AB.

$$AB = \begin{pmatrix} 2 & 3 \\ \boxed{1 \quad -5} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 4 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 3 \\ -2 \end{matrix}} & \boxed{\begin{matrix} 6 \\ 3 \end{matrix}} \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ \boxed{? \quad ? \quad ?} \end{pmatrix}$$

The second row is $\boxed{(-1 \quad 13 \quad -9)}$

The second row is just

$$(1 \quad -5) \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

Likewise, the first row is

$$(2 \quad 3) \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

In fact, if $\vec{r}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{r}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, then

$$A = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \end{pmatrix} B = \begin{pmatrix} \vec{r}_1^{\,T} B \\ \vec{r}_2^{\,T} B \end{pmatrix}$$

In general, if

$$A = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{pmatrix} \quad \text{and} \quad B = (\vec{b}_1 \cdots \vec{b}_p),$$

$(\vec{r}_i \in \mathbb{R}^n$ and $\vec{b}_j \in \mathbb{R}^n)$

then $$AB = (A\vec{b}_1 \cdots A\vec{b}_p)$$

and $$AB = \begin{pmatrix} \vec{r}_1^{\,T} B \\ \vdots \\ \vec{r}_m^{\,T} B \end{pmatrix}.$$

<u>Expl 6</u>

$$A = \begin{pmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{pmatrix}$$

Find the <u>second row of AB.</u>

$$\underset{1\times 3}{(-1 \;\; 3 \;\; -4)} \underset{3\times 2}{\begin{pmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{pmatrix}} = (-4+21-12 \quad 6+3-8)$$

$$= \underset{1\times 2}{\boxed{(5 \quad 1)}}$$