1.9 The matrix of a linear transformation

standard basis vectors of $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

<u>Thm 10</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let

$$A = \big(\underbrace{T(\vec{e}_1)}_{\text{vector in } \mathbb{R}^m} \; T(\vec{e}_2) \cdots T(\vec{e}_n) \big) \leftarrow \begin{matrix} m \times n \\ \text{matrix} \end{matrix}$$

Then $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Moreover, A is the unique matrix with this property.

<u>Pf:</u> Let $\vec{x} \in \mathbb{R}^n$ be arbitrary. Then

$$\vec{x} = I_n \vec{x} = (\vec{e}_1 \, \vec{e}_2 \, \vec{e}_3 \cdots \vec{e}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n .$$

Therefore,

$$T(\vec{x}) = T(x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n)$$

$$= x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n)$$

$$= \left(T(\vec{e}_1) \cdots T(\vec{e}_n) \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\vec{x}.$$

Proving the uniqueness of A is Exer. 1.9.41 (unassigned). $\square$

The matrix A in Thm 10 is called the <u>standard matrix for the linear transformation T</u> (or just the <u>matrix for T</u>).

<u>Expl 1</u>

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be linear with

$$T(\vec{e}_1) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}, \quad T(\vec{e}_2) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}.$$

<u>Find the matrix for T.</u>

By Thm 10, the matrix for T is

$$A = \left(T(\vec{e}_1) \; T(\vec{e}_2) \right) = \boxed{\begin{pmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{pmatrix}}$$

<u>Expl 2</u>

Let T be the dilation transformation on $\mathbb{R}^2$, $T(\vec{x}) = 3\vec{x}$. <u>Find the matrix for T.</u>

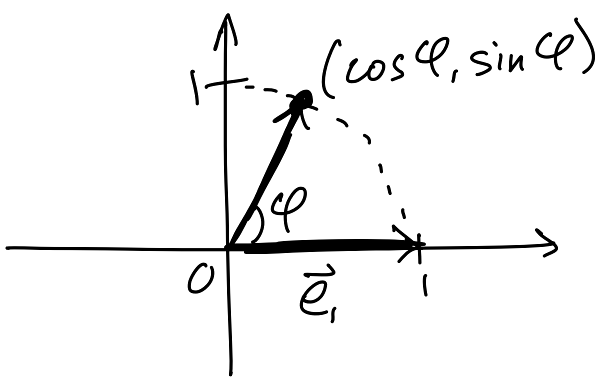
$$T(\vec{e}_1) = 3\vec{e}_1 = 3\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

$$T(\vec{e}_2) = 3\vec{e}_2 = 3\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}$$

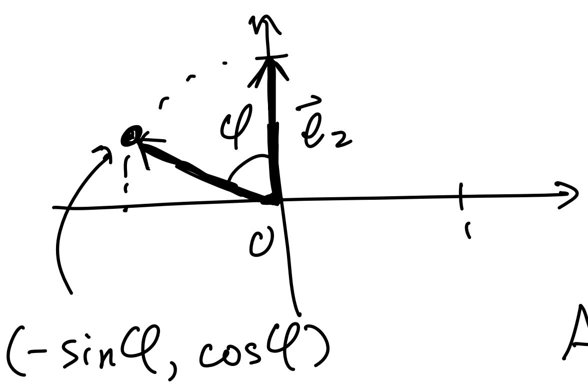
$$A = (T(\vec{e}_1)\ T(\vec{e}_2)) = \boxed{\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}}$$

Expl 3

Let φ be a real number. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector counter-clockwise by φ radians. The transformation T is linear. Find the matrix of T.



$$T(\vec{e}_1) = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$$



$$T(\vec{e}_2) = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

$$A = (T(\vec{e}_1)\ T(\vec{e}_2)) = \boxed{\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}}$$

[Look at Tables 1-4 on p. 78]

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if

the range of T is $\mathbb{R}^m$

(OR for all $\vec{b} \in \mathbb{R}^m$, there exists at least one $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$)

existence question

surjective is a synonym for "onto"

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if

$T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$

(OR $\vec{x} \neq \vec{y}$ implies $T(\vec{x}) \neq T(\vec{y})$)

(OR for all $\vec{b} \in \mathbb{R}^m$, there is at most one $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$)

uniqueness question

injective is a synonym for "one-to-one"

bijective means "one-to-one and onto"

Expl Let T be the linear transformation w/ matrix

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Is T onto? Is T one-to-one?

$T(\vec{x}) = \vec{b}$ is the same as $A\vec{x} = \vec{b}$

For every $\vec{b}$, $A\vec{x} = \vec{b}$ is consistent (no row of the form $(0 \cdots 0 \; b^{\neq 0})$ in augmented matrix), so at least one soln

$\boxed{T \text{ is onto}}$

For every $\vec{b}$, $A\vec{x} = \vec{b}$ has a free variable (x_3), so there would be infinitely many solutions

$\boxed{T \text{ is not one-to-one}}$

<u>Thm 11:</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then T is one-to-one iff $\vec{x} = \vec{0}$ is the only soln to $T(\vec{x}) = \vec{0}$. (i.e. $A\vec{x} = \vec{0}$ has only one soln)

<u>Pf:</u> First note that $T(\vec{0}) = \vec{0}$ because T is linear. Now, assume T is one-to-one. This means that if $\vec{x} \neq \vec{0}$, then $T(\vec{x}) \neq T(\vec{0})$ or $T(\vec{x}) \neq \vec{0}$. So $\vec{x} = \vec{0}$ is the only soln to $T(\vec{x}) = \vec{0}$.

For the converse, assume $\vec{x} = \vec{0}$ is the only soln to $T(\vec{x}) = \vec{0}$. Assume $T(\vec{u}) = T(\vec{v})$ for some $\vec{u}, \vec{v} \in \mathbb{R}^n$. Since T is linear,

$$T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{0}.$$

Thus, $\vec{u}-\vec{v}=\vec{0}$, and so $\vec{u}=\vec{v}$. This shows that T is one-to-one. $\square$

<u>Thm 12</u> Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be linear w/ matrix A.

(a) T is onto iff the cols. of A span $\mathbb{R}^m$

(b) T is one-to-one iff the cols. of A are lin. indep.

<u>Pf:</u>

(a) Let U be the range of T and write $A=(\vec{a}_1 \cdots \vec{a}_n)$. For any vector $\vec{b}\in\mathbb{R}^m$,

$\vec{b}\in U$ iff there exists $x\in\mathbb{R}^n$ w/ $T(\vec{x})=\vec{b}$
iff " $A\vec{x}=\vec{b}$
iff " $x_1\vec{a}_1+\cdots+x_n\vec{a}_n=\vec{b}$
iff $\vec{b}\in \mathrm{Span}\{\vec{a}_1,\dots,\vec{a}_n\}$.

Thus, T is onto iff $U=\mathbb{R}^m$ iff $\mathrm{Span}\{\vec{a}_1,\dots,\vec{a}_n\}=\mathbb{R}^m$.

$\mathrm{Span}\{\vec{a}_1,\dots,\vec{a}_n\}=$ range of T

(b)

T is one-to-one iff $\vec{x}=\vec{0}$ is the only soln to $T(\vec{x})=\vec{0}$
iff " $A\vec{x}=\vec{0}$
iff $x_1=x_2=\cdots=x_n=0$ is the only soln to $x_1\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n=\vec{0}$

iff $\{\vec{a}_1, \dots, \vec{a}_n\}$ are lin. indep. $\square$

Expl 5

Let $T(x,y) = (3x+y,\ 5x+7y,\ x+3y)$. Show that T is a one-to-one linear transformation. Is T onto?

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix},\ T(\vec{x}) = \begin{pmatrix} 3x+y \\ 5x+7y \\ x+3y \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$T(\vec{x}) = A\vec{x}$ for all $\vec{x}$

So $\boxed{T \text{ is linear}}$ ✓

Cols of A: $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$

not multiples, so lin. indep.

$\therefore$ $\boxed{T \text{ is one-to-one}}$ ✓

range of $T = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} \right\}$.

Thm Let A be an $m \times n$ matrix. If $m > n$, then the cols. of A do not span $\mathbb{R}^m$.

Pf: By Thm 4 (Sect. 1.4), the cols. of A span $\mathbb{R}^m$ iff A has a pivot position in every row. But A has only n columns, so can have at most n pivot positions. And yet there are $m > n$ rows, so this is impossible.

Since A is 3×2 and $3 > 2$, the cols. of A don't span $\mathbb{R}^3$, and $\boxed{T \text{ is not onto}}$