1.8 Introduction to linear transformations

A <u>transformation</u> T from $\mathbb{R}^n$ to $\mathbb{R}^m$

is a function $T: \mathbb{R}^n \to \mathbb{R}^m$

domain of T — codomain of T

$T(\vec{x}) \leftarrow$ image of $\vec{x}$ under T

$\{T(\vec{x}) : \vec{x} \in \mathbb{R}^n\} \leftarrow$ range of T

If A is an $m \times n$ matrix, then

$A\vec{x}$ makes sense whenever $\vec{x} \in \mathbb{R}^n$

and $A\vec{x}$ is a vector in $\mathbb{R}^m$.

$T(\vec{x}) = A\vec{x}$ is a transformation

from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Any transformation of this type is called

a matrix transformation.

Expl 1

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}, \quad T(\vec{x}) = A\vec{x}$$

A is 3×2, $\quad T: \mathbb{R}^2 \to \mathbb{R}^3$

(a) Find the image of $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ under T.

(b) Find an $\vec{x} \in \mathbb{R}^2$ whose image is $\vec{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$.

(c) Is there more than one $\vec{x} \in \mathbb{R}^2$ whose image is $\vec{b}$?

(d) Is $\vec{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ in the range of T?

(a) $$T(\vec{u}) = A\vec{u} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}}$$

(b) $T(\vec{x}) = \vec{b}$ iff $A\vec{x} = \vec{b}$

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right)$$

$x_2 = -\frac{1}{2}$

$x_1 - 3x_2 = 3$

$x_1 = 3(-\frac{1}{2}) + 3 = \frac{3}{2}$

$$\boxed{\vec{x} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}}$$

(c) $\boxed{\text{No. solution in (b) is unique}}$

(d) Does $T(\vec{x}) = \vec{c}$ have a soln?

$$A\vec{x} = \vec{c}$$

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 10 \end{array}\right) \quad \text{No soln}$$

$\boxed{\text{No, } \vec{c} \text{ is not in range of } T}$

<u>Expl 2</u> Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x,y,z) = (x, y, 0)$. Is T a matrix transformation? (T projects $\mathbb{R}^3$ onto the xy-plane)

In our notation,

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Want a 3×3 matrix A such that $T(\vec{x}) = A\vec{x}$

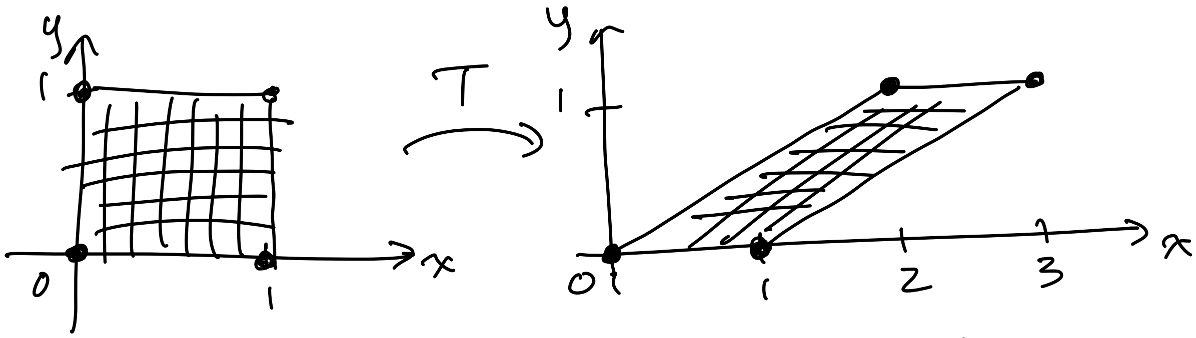
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\boxed{\text{Yes, } T(\vec{x}) = A\vec{x}, \text{ where } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.}$$

Expl 3

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$. Find a formula for $T(x,y)$.

$$\boxed{T(x,y) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ y \end{pmatrix}}$$



$$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Called a <u>shear transformation</u>

A transformation is <u>linear</u> if

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall \vec{u}, \vec{v}$

(ii) $T(c\vec{u}) = c\,T(\vec{u}) \quad \forall c, \vec{u}$

Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation. (Will see that in Section 1.9.)

And every matrix transformation is linear

<u>Thm</u> If T is a linear transformation, then

(a) $T(\vec{0}) = \vec{0}$

(b) $T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$, and

(c) $T(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p)$.

<u>Pf:</u> (a) $T(\vec{0}) = T(0\cdot\vec{0}) = 0\cdot T(\vec{0}) = \vec{0}$

(b) $T(c\vec{u}+d\vec{v}) = T(c\vec{u}) + T(d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.

(c) Use (b) multiple times. $\square$

<u>Expl 4</u> Let $r \in \mathbb{R}$ and define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$. Show that T is linear.

($0 \le r \le 1 \to T$ is a <u>contraction</u>

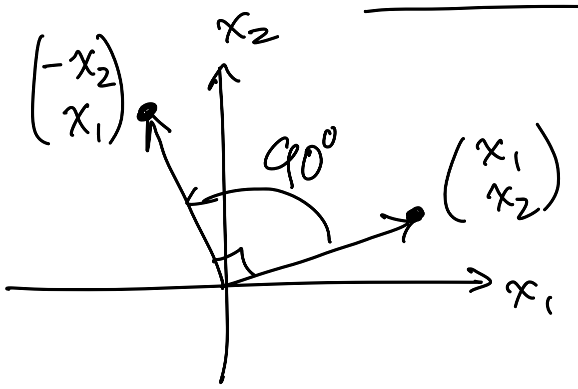
$r > 1 \to T$ is a <u>dilation</u>)

$T(\vec{u}+\vec{v}) = r(\vec{u}+\vec{v}) = r\vec{u} + r\vec{v} = T(\vec{u}) + T(\vec{v})$ ✓

$T(c\vec{u}) = r \cdot c\vec{u} = c \cdot r\vec{u} = cT(\vec{u})$ ✓

Expl 5

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ act on a vector in $\mathbb{R}^2$ by rotating it $90°$ counter-clockwise. Show that T is a matrix transformation. Find $T(\vec{u})$, $T(\vec{v})$, and $T(\vec{u}+\vec{v})$, where $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.



T maps $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$T(\vec{x}) = A\vec{x}$

$T(\vec{u}) = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $T(\vec{v}) = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$.