1.7 Linear independence

despite the notation, we allow duplicates

A set $\{\vec{v}_1, \dots, \vec{v}_p\} \subset \mathbb{R}^n$ is linearly dependent if there are constants $c_1, \dots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

$\leftarrow$ a linear dependence relation

The set is linearly independent otherwise.

Expl 1

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. indep? If not, find a lin. dep. relation.

Want a nontrivial soln to

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$\underbrace{(\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3)}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \vec{0}$$

$$A\vec{x} = \vec{0}$$

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_2 \to R_3 \end{matrix} \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{pmatrix}$$

$$R_3 - 2R_2 \to R_3 \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3 is a free var.
there are nontrivial solns

$\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ are lin. dep.}}$

$$\begin{matrix} -\frac{1}{3}R_2 \to R_1 \\ R_1 - 4R_2 \to R_1 \end{matrix} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 2x_3 = 0 \to x_1 = 2x_3$
$x_2 + x_3 = 0 \to x_2 = -x_3$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Any soln will do (other than $x_3 = 0$).

Take $x_3 = 1$: $x_1 = 2$, $x_2 = -1$, $x_3 = 1$

$$\boxed{2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}}$$

$$A = (\vec{a}_1 \cdots \vec{a}_n)$$

$A\vec{x} = \vec{0}$ has only the trivial soln $\vec{x} = \vec{0}$

iff

$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0}$ has only the trivial soln
$x_1 = \cdots = x_n = 0$

iff

$\{\vec{a}_1, \dots, \vec{a}_n\}$ are lin. indep.

$\boxed{\text{Cols of } A \text{ are lin. indep. iff } A\vec{x} = \vec{0} \text{ has only the trivial soln.}}$

Expl 2

$$A = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix}$$

Are the cols. of A lin. indep?

$$\left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \leftrightarrow R_2 \\ R_3 - 5R_1 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right)$$

$$R_3 + 2R_2 \to R_3 \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right)$$

no free vars.
$\vec{x} = \vec{0}$ only soln

$\boxed{\text{YES}}$

Thm $\{\vec{v}\}$ is lin. dep. iff $\vec{v} = \vec{0}$.

Pf: Suppose $\{\vec{v}\}$ is lin. dep. Then $\exists c \neq 0$ s.t. $c\vec{v} = \vec{0}$. Since $c \neq 0$, $\frac{1}{c}$ is a number, and we may write $\vec{v} = \frac{1}{c} c\vec{v} = \frac{1}{c}\vec{0} = \vec{0}$.

Now suppose $\vec{v} = \vec{0}$. Then $1\,\vec{v} = \vec{0}$ is a lin. dep. relation, so $\{\vec{v}\}$ is lin. dep. $\square$

Thm $\{\vec{v}_1, \vec{v}_2\}$ is lin. dep. iff one vector is a multiple of the other.

Pf: Suppose $\{\vec{v}_1, \vec{v}_2\}$ is lin. dep. Let $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ be a lin. dep. relation. Then either $c_1 \neq 0$ or $c_2 \neq 0$. If $c_1 \neq 0$, then $\vec{v}_1 = \frac{c_2}{c_1}\vec{v}_1$. If $c_2 \neq 0$, then $\vec{v}_2 = \frac{c_1}{c_2}\vec{v}_1$.

Now suppose one vector is a multiple of the other. If $\vec{v}_1 = c\vec{v}_2$, then $1\vec{v}_1 - c\vec{v}_2 = \vec{0}$ is a lin. dep. relation. If $\vec{v}_2 = c\vec{v}_1$, then $-c\vec{v}_1 + 1\vec{v}_2 = \vec{0}$ is a lin. dep. relation. Thus, $\{\vec{v}_1, \vec{v}_2\}$ is lin. dep. $\square$

c's here might be 0

Thm 7 (characterization of lin. dep. sets)

$S = \{\vec{v}_1, \dots, \vec{v}_p\}$ is lin. dep. iff at least one vector in S can be written as a linear combination of the other vectors in S.

In fact, if S is lin. dep. and $\vec{v}_1 \neq \vec{0}$, then $\exists\, j > 1$ such that $\vec{v}_j$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_{j-1}$.

Pf: Suppose that at least one vector in S can be written as a lin. combo. of the others. So there is some k with $1 \le k \le p$ and

$$\vec{v}_k = c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_{k+1}\vec{v}_{k+1} + \cdots + c_p\vec{v}_p.$$

(Any or all of these c's might be 0.) This gives

$$c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + (-1)\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_p\vec{v}_p = \vec{0}$$

Since the coeff. of $\vec{v}_k$ is nonzero, this is a lin. dep. relation, so $\{\vec{v}_1, \dots, \vec{v}_p\}$ is lin dep.

Now suppose S is lin. dep.

<u>Case 1</u>: <u>$\vec{v}_1 = \vec{0}$</u>. In this case,

$$\vec{v}_1 = 0\vec{v}_2 + 0\vec{v}_3 + \cdots + 0\vec{v}_p,$$

and the proof is done.

<u>Case 2</u>: <u>$\vec{v}_1 \ne \vec{0}$</u>. Let

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = 0$$

be a lin. dep. relation, so that at least one c_k is nonzero. Let $j = \max\{k : c_k \ne 0\}$.

Then $c_k = 0 \ \forall k > j$, so the above becomes

$$c_1\vec{v}_1 + \cdots + c_j\vec{v}_j = \vec{0}.$$

Since $c_j \ne 0$, we can write

$$\vec{v}_j = \left(-\frac{c_1}{c_j}\right)\vec{v}_1 + \left(-\frac{c_2}{c_j}\right)\vec{v}_2 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right)\vec{v}_{j-1},$$

showing that $\vec{v}_j$ is a lin. combo. of $\vec{v}_1, \dots, \vec{v}_{j-1}$. $\square$

Expl 3a

Are $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ lin. indep?

$\vec{v}_2 = 2\vec{v}_1$. $\boxed{\text{No, they are lin. dep.}}$

Expl 3b

Are $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ lin. indep?

Neither is a mult. of the other, so

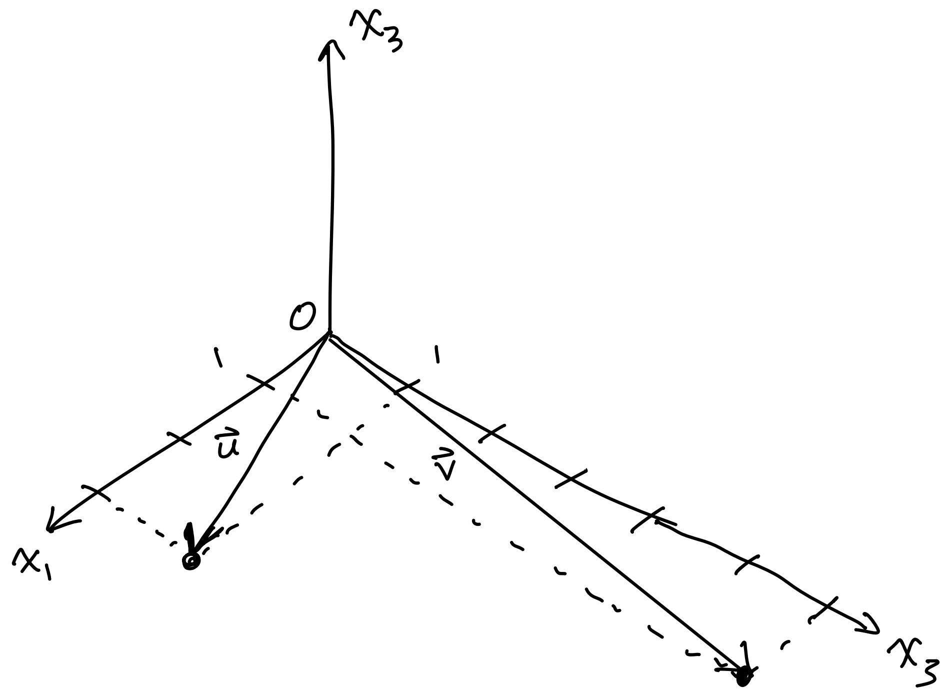
$\boxed{\text{Yes, they are lin. indep.}}$

Expl 4

$$\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$$

Describe Span$\{\vec{u}, \vec{v}\}$ geometrically.

Explain why $\vec{w} \in \text{Span}\{\vec{u}, \vec{v}\}$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ is lin. dep.



Span$\{\vec{u}, \vec{v}\}$ is the x_1x_2-plane

$\vec{u} \neq \vec{0}$, so $\{\vec{u}, \vec{v}, \vec{w}\}$ is lin. dep. iff

$\vec{v} = c\vec{u}$ or $\vec{w} = c_1\vec{u} + c_2\vec{v}$ (Thm. 7)

But $\vec{v} \neq c\vec{u}$.

So $\{\vec{u}, \vec{v}, \vec{w}\}$ is lin. dep. iff

$\vec{w} = c_1\vec{u} + c_2\vec{v}$ iff

$\vec{w} \in \text{Span}\{\vec{u}, \vec{v}\}$.

Thm 8 Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. If $p > n$, then S is lin. dep.

Thm 8 (matrix version) Let A be an $n \times p$ matrix. If $p > n$, then the cols. of A are lin. dep. (Converse not true!)

Pf: Write $A = (\vec{v}_1 \cdots \vec{v}_p)$. The eqn. $A\vec{x} = \vec{0}$ has augmented matrix $(\vec{v}_1 \cdots \vec{v}_p \mid \vec{0})$. Since the number of rows is n and $n < p$, there must be at least one free variable. Thus, $A\vec{x} = \vec{0}$ has nontrivial solns, so $\vec{v}_1, \dots, \vec{v}_p$ are lin. dep. □

Expl 5 Are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ lin. dep. or lin. indep.?

There are 3 vectors in $\mathbb{R}^2$ and $3 > 2$, so they are lin. dep.

Thm 9 If $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ contains the zero vector, then S is lin. dep.

Pf: Suppose there is a k with $1 \leq k \leq p$ and $\vec{v}_k = \vec{0}$. Then

$$0\vec{v}_1 + \cdots + 0\vec{v}_{k-1} + 1\vec{v}_k + 0\vec{v}_{k+1} + \cdots + 0\vec{v}_p = \vec{0}$$

is a lin. dep. relation, so S is lin. dep. □

Expl 6a

Are $\begin{pmatrix}1\\7\\6\end{pmatrix}, \begin{pmatrix}2\\0\\9\end{pmatrix}, \begin{pmatrix}3\\1\\5\end{pmatrix}, \begin{pmatrix}4\\1\\8\end{pmatrix}$ lin. dep?

4 vectors in $\mathbb{R}^3$, $4>3$, so $\boxed{\text{Yes, lin. dep.}}$

Expl 6b

Are $\begin{pmatrix}2\\3\\5\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\8\end{pmatrix}$ lin. dep.?

Contains $\vec{0}$, so $\boxed{\text{Yes, lin. dep.}}$

Expl 6c

Are $\begin{pmatrix}-2\\4\\6\\10\end{pmatrix}, \begin{pmatrix}3\\-6\\-9\\15\end{pmatrix}$ lin. dep?

Only 2 vectors. Neither is a multiple of the other (no common ratio; $-\frac{3}{2}$ for three entries, $\frac{3}{2}$ for the fourth). $\boxed{\text{No, lin. indep.}}$