1.5 Solution sets of linear systems

$\underset{m\times n}{A}\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ is <u>homogeneous</u> if $\vec{b} = \vec{0}$

$A\vec{x} = \underset{m\times 1}{\vec{0}}$ always has $\underbrace{\vec{x} = \underset{n\times 1}{\vec{0}}}$ as a soln

the <u>trivial solution</u>

Any other solutions (if they exist) are called <u>nontrivial solutions</u>

$$\boxed{\text{$A\vec{x} = \vec{0}$ has a nontrivial soln iff the eqn has at least one free variable.}}$$

<u>Expl 1</u>

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Does this system have a nontrivial soln?
Describe the solution set.

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equiv to $A\vec{x} = \vec{0}$, $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$

aug. matrix : $(A|\vec{0})$

$$= \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right) \begin{array}{l} R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right)$$

$$R_3 + 3R_2 \to R_3 \quad \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 is a free variable, so there are nontrivial solutions

$$\begin{array}{l} \frac{1}{3}R_2 \to R_2 \\ R_1 - 5R_2 \to R_1 \end{array} \quad \left(\begin{array}{ccc|c} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \frac{1}{3}R_1 \to R_1 \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - \frac{4}{3}x_3 = 0 \longrightarrow x_1 = \frac{4}{3}x_3$$

$$\cancel{4/3}$$

$$x_2 = 0$$

$$0 = 0$$

rename to t

call this $\vec{v}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\text{soln set} = \underbrace{\{t\vec{v} : t \in \mathbb{R}\}}_{\text{line through the origin parallel to } \vec{v}}, \text{ where } \vec{v} = \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}}$$

Expl 3

same A as before

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

Find the soln set of $A\vec{x} = \vec{b}$. not homogeneous

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \begin{array}{l} R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right)$$

$$R_3 + 3R_2 \to R_3 \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \frac{1}{3}R_2 \to R_2 \\ R_1 - 5R_2 \to R_1 \end{array} \left(\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\frac{1}{3}R_1 \to R_1 \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{rl} x_1 \quad - \frac{4}{3}x_3 = -1 & \to x_1 = \frac{4}{3}x_3 - 1 \\ x_2 \quad = 2 & \\ 0 = 0 & \end{array}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{\text{call this } \vec{p}} + \underbrace{x_3}_{\text{rename to } t} \underbrace{\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}}_{\text{same } \vec{v} \text{ as before}}$$

$$\text{Soln set} = \underbrace{\{\vec{p} + t\vec{v} : t \in \mathbb{R}\}}_{\substack{\text{line through the point } \vec{p}, \\ \text{parallel to the vector } \vec{v}}}$$