1.4 The matrix equation $A\vec{x} = \vec{b}$

must match

If A is an $m \times n$ matrix and
$\vec{x}$ is an $n \times 1$ matrix (a vector in $\mathbb{R}^n$),

then $A\vec{x}$ is defined as follows:

write $A = (\underbrace{\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n}_{\text{each } \vec{a}_j \in \mathbb{R}^m})$

and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$A\vec{x} = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$:= \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}_{\substack{\text{a vector in } \mathbb{R}^m \\ \text{(an } m\times 1 \text{ matrix)}}}$$

Can also calculate like this:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Expls 1&5

(a) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -5 \end{pmatrix} + 7\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ -15 \end{pmatrix} + \begin{pmatrix} -7 \\ 21 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

OR $= \begin{pmatrix} 1(4) + 2(3) + (-1)(7) \\ 0(4) + (-5)(3) + 3(7) \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 2(4) - 3(7) \\ 8(4) - 0(7) \\ -5(4) + 2(7) \end{pmatrix} = \begin{pmatrix} 8 - 21 \\ 32 \\ -20 + 14 \end{pmatrix} = \begin{pmatrix} -13 \\ 32 \\ -6 \end{pmatrix}$

OR $= 4\begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} + 7\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 32 \\ -20 \end{pmatrix} + \begin{pmatrix} -21 \\ 0 \\ 14 \end{pmatrix} = \begin{pmatrix} -13 \\ 32 \\ -6 \end{pmatrix}$

(c) $\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{identity matrix}}\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1\cdot r + 0\cdot s + 0\cdot t \\ 0\cdot r + 1\cdot s + 0\cdot t \\ 0\cdot r + 0\cdot s + 1\cdot t \end{pmatrix}$

$= \begin{pmatrix} r \\ s \\ t \end{pmatrix}$

$$I_n = \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{n \times n}, \quad I_n \vec{x} = \vec{x}, \text{ for all } \vec{x} \in \mathbb{R}^n$$

Properties of the matrix-vector product:

(a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

(b) $A(c\vec{u}) = c(A\vec{u})$

<u>Proof of (a):</u>

$$A(\vec{u} + \vec{v}) = (\vec{a}_1 \cdots \vec{a}_n) \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

$$= (u_1 + v_1)\vec{a}_1 + \cdots + (u_n + v_n)\vec{a}_n$$

$$= (u_1\vec{a}_1 + \cdots + u_n\vec{a}_n) + (v_1\vec{a}_1 + \cdots + v_n\vec{a}_n)$$

$$= A\vec{u} + A\vec{v}.$$

<u>Proof of (b)</u>

$$A(c\vec{u}) = (\vec{a}_1 \cdots \vec{a}_n) \begin{pmatrix} cu_1 \\ \vdots \\ cu_n \end{pmatrix}$$

$$= cu_1\vec{a}_1 + \cdots + cu_n\vec{a}_n$$

$$= c(u_1\vec{a}_1 + \cdots + u_n\vec{a}_n) = c(A\vec{u})$$

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linear system:

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix} \longrightarrow$$

augmented matrix

$$(\vec{a}_1 \cdots \vec{a}_n \mid \vec{b})$$

$$\updownarrow$$

vector eqn:

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

$$\updownarrow$$

matrix eqn:

$$A\vec{x} = \vec{b}$$

Expl 3 Let $A = \begin{pmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{pmatrix}$. Is

$A\vec{x} = \vec{b}$ consistent for all $\vec{b}$?

Let's figure out which $\vec{b}$'s make $A\vec{x} = \vec{b}$ consistent.

augmented matrix:

$$\left(\begin{array}{cccc} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right) \begin{array}{l} R_2+4R_1 \to R_2 \\ R_3+3R_1 \to R_3 \end{array} \left(\begin{array}{cccc} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 7 & 5 & 3b_1+b_3 \end{array}\right)$$

$$R_3 - \tfrac{1}{2}R_2 \to R_3 \left(\begin{array}{cccc} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{array}\right)$$

$$3b_1 + b_3 - \tfrac{1}{2}(4b_1 + b_2) = 3b_1 + b_3 - 2b_1 - \tfrac{1}{2}b_2 = b_1 - \tfrac{1}{2}b_2 + b_3$$

Inconsistent whenever $b_1 - \frac{1}{2}b_2 + b_3 \neq 0$

e.g. inconsistent for $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$\boxed{\text{No, not consistent for all } \vec{b}}$

<u>Thm 4</u> Let A be an $m \times n$ matrix.

Then TFAE (the following are equivalent)

(a) $\forall \vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a soln

(b) Every $\vec{b} \in \mathbb{R}^m$ is a lin. combo of columns of A

(c) The columns of A span $\mathbb{R}^m$
(i.e. $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^m$)

(d) A has a pivot in every row

all are true or all are false