# 1.3 Vector equations

A (column) vector is a matrix with only one column.

$\mathbb{R}^n$ = the set of $n \times 1$ matrices

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad c \in \mathbb{R}$$

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}, \quad c\vec{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix}$$

vector addition

scalar multiplication

**Expl 1**

$$\vec{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Find $4\vec{u}$, $(-3)\vec{v}$, and $4\vec{u} + (-3)\vec{v}$.

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$$4\vec{u} = \begin{pmatrix} 4(1) \\ 4(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}, \quad (-3)\vec{v} = \begin{pmatrix} (-3)(2) \\ (-3)(-5) \end{pmatrix} = \begin{pmatrix} -6 \\ 15 \end{pmatrix}$$

$$4\vec{u} + (-3)\vec{v} = \begin{pmatrix} 4 \\ -8 \end{pmatrix} + \begin{pmatrix} -6 \\ 15 \end{pmatrix} = \begin{pmatrix} 4-6 \\ -8+15 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

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Subtraction: $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$

e.g. $4\vec{u} - 3\vec{v} = 4\vec{u} + (-1)(3\vec{v})$

$= 4\vec{u} + ((-1)(3))\vec{v}$

$= 4\vec{u} + (-3)\vec{v}$

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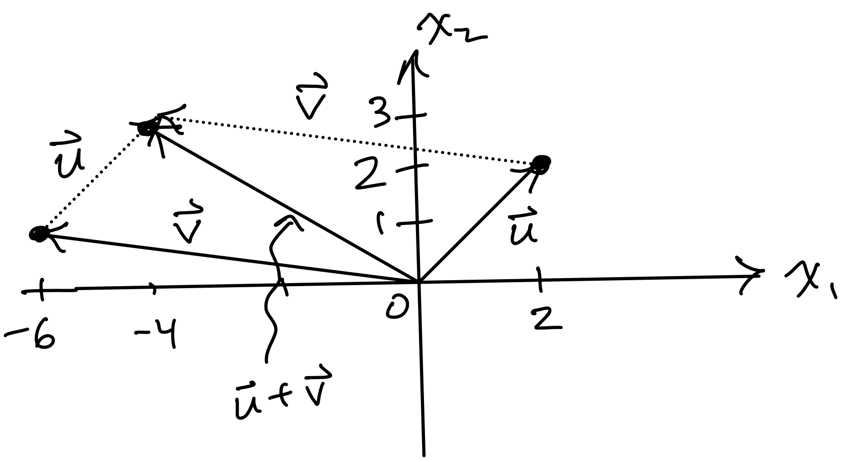
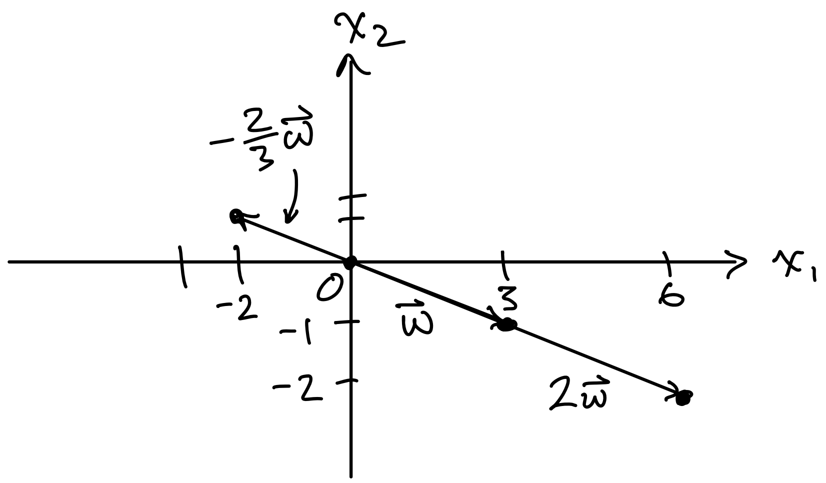
Vectors in $\mathbb{R}^2$ can be visualized as points in the plane, or arrows from the origin to a point.

$\vec{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

$\vec{u} + \vec{v} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$




$\vec{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $2\vec{w} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, $-\frac{2}{3}\vec{w} = \begin{pmatrix} -2 \\ \frac{2}{3} \end{pmatrix}$



$\{t\vec{w} : t \geq 0\}$ is a ray starting at the origin

$\{t\vec{w} : t \in \mathbb{R}\}$ is a line through the origin

Vectors in $\mathbb{R}^3$ have a similar geometric interpretation.

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## Properties of $\mathbb{R}^n$

vector of all 0's

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(iii) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

(iv) $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$

$(-1)\vec{u}$

(v) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(vi) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

(vii) $(cd)\vec{u} = c(d\vec{u})$

(viii) $1\vec{u} = \vec{u}$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$ , $c_1, c_2, \ldots, c_p \in \mathbb{R}$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \leftarrow$$

linear combination of $\vec{v}_1, \ldots, \vec{v}_p$

weights

## Expl 5

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$$

Can $\vec{b}$ be written as a linear combination of $\vec{a}_1$ and $\vec{a}_2$? If so, find weights $c_1$ and $c_2$

so that $\vec{b} = c_1\vec{a}_1 + c_2\vec{a}_2$.

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Change $c_1, c_2$ to $x_1, x_2$ so it looks more familiar:

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$$

$$x_1\begin{pmatrix}1\\-2\\-5\end{pmatrix} + x_2\begin{pmatrix}2\\5\\6\end{pmatrix} = \begin{pmatrix}7\\4\\-3\end{pmatrix}$$

$$\begin{pmatrix}x_1+2x_2\\-2x_1+5x_2\\-5x_1+6x_2\end{pmatrix} = \begin{pmatrix}7\\4\\-3\end{pmatrix} \iff \underbrace{\begin{aligned}x_1+2x_2&=7\\-2x_1+5x_2&=4\\-5x_1+6x_2&=-3\end{aligned}}_{\text{Question is: does this system have a soln? If so, find one}}$$

augmented matrix:

$$\begin{array}{ccc} \left(\begin{array}{cc|c}1&2&7\\-2&5&4\\-5&6&-3\end{array}\right) & = & (\vec{a}_1\ \vec{a}_2 \mid \vec{b}) \\ \uparrow\ \ \uparrow\ \ \ \uparrow & & \\ \vec{a}_1\ \ \vec{a}_2\ \ \ \vec{b} & & \end{array}$$

In the future, can skip straight to this step.

$\underset{\text{row reduce}}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{l} x_1 = 3 \\ x_2 = 2 \end{array}$

$$\boxed{\vec{b} = 3\vec{a}_1 + 2\vec{a}_2}$$

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Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ = set of all lin. combos. of $\vec{v}_1, \dots, \vec{v}_p$

= all vectors of the form

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$$

also called the <u>subset of $\mathbb{R}^n$ spanned</u> (or <u>generated</u>) <u>by $\vec{v}_1, \dots, \vec{v}_p$</u>

<u>Expl 6</u>

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix}$$

Is $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2\}$? (Geometrically, $\text{Span}\{\vec{a}_1, \vec{a}_2\}$ is a plane in $\mathbb{R}^3$ that passes through the origin.)

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$$(\vec{a}_1\ \vec{a}_2 \mid \vec{b}) = \left(\begin{array}{cc|c} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array}\right) \underset{\text{row reduce}}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{array}\right)$$

Last row shows the system $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ is inconsistent.

So, $\boxed{\text{no, } \vec{b} \notin \text{Span}\{\vec{a}_1, \vec{a}_2\}}$