1.2 Row reduction and echelon form

A nonzero row of a matrix is a row with at least one entry that is not zero.

The leading entry of a nonzero row is the leftmost nonzero entry.

A matrix is in (row) echelon form (or REF) if:

(1) All nonzero rows are above the rows of zeros.

(2) Each leading entry is in a column to the right of the leading entry of the row above it.

A matrix is in reduced (row) echelon form (RREF) if (1) and (2) are true, and also:

(4)* Every leading entry is 1.

(5) Each leading 1 is the only nonzero entry in its column.

* There is a (3) in the book, but it's redundant.

An echelon matrix is one that's in REF.

A reduced echelon matrix is one that's in RREF.

Expl 1

$$\begin{pmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{pmatrix}$$

echelon form $\left(\begin{array}{l} *\text{'s can be anything} \\ \blacksquare\text{'s must be nonzero} \end{array} \right)$

- The locations of the squares are called pivot positions.
- The colums with pivots are called pivot columns.

Replace $\blacksquare$'s with 1's and put all 0's above the 1's to get reduced echelon form

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To row reduce a matrix to echelon form is to perform elementary row operations on it until it is in echelon form.

For a given matrix A, there are many matrices in echelon form that it can be row reduced to, but only one in reduced echelon form.

Thm Each matrix is row equivalent to one and only one reduced echelon matrix.

Row reduction algorithm

forward phase

1. Find the leftmost nonzero column.
   This is a pivot column.
   The pivot position is at the top.
2. Select a nonzero entry in the pivot column.
   Use this entry as a pivot.
   If necessary, move it into pivot position.
3. Use row replacement to create 0's below the pivot.
4. Cover the row with the pivot and all rows above it.
   Apply 1-3 to the resulting submatrix.
   Repeat until reaching echelon form.

backward phase

5. Beginning with the rightmost pivot and working upward and to the left, use scaling and row replacement to change all pivots to 1 and put 0's above all pivots.
   At the end, you reduced echelon form.

Expl 2

$$A = \begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}$$

Row reduce A to echelon form.

What are the pivot columns of A?

Only need steps 1-4

$$\xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix} \xrightarrow[R_3+2R_1 \to R_3]{R_2+R_1 \to R_2} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$$\xrightarrow[R_4+\frac{3}{2}R_2 \to R_4]{R_3-\frac{5}{2}R_2 \to R_3} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \boxed{\begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

pivots

$\boxed{\text{Pivot columns are } 1, 2, 4}$

Expl 3

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

Row reduce A to reduced echelon form.

$$R_1 \leftrightarrow R_3 \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$R_2 - R_1 \to R_2 \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$R_3 - \tfrac{3}{2} R_2 \to R_3 \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

pivots ↖ start here

$$\begin{array}{l} R_1 - 6R_3 \to R_1 \\ R_2 - 2R_3 \to R_2 \end{array} \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\tfrac{1}{2} R_2 \to R_2 \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$R_1 + 9R_2 \to R_1 \quad \begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\tfrac{1}{3} R_1 \to R_1 \quad \boxed{\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}}$$

Expl

A linear system's augment matrix has been row reduced to this reduced echelon matrix:

$$\begin{pmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the solution set of the linear system.

new augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

new system

$$\begin{aligned} x_1 \quad\quad - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

pivot columns $\longleftrightarrow$ basic variables

x_3 is not a basic var., it's a free variable

Solve:

$$\boxed{\begin{aligned} x_1 &= 5x_3 + 1 \\ x_2 &= -x_3 + 4 \\ x_3 &= \text{any} \end{aligned}}$$

solution set

OR

$$\boxed{\{(5x_3+1, -x_3+4, x_3) : x_3 \in \mathbb{R}\}}$$

parametric descriptions

Expl 5 Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

We are being asked which of these 3 hold:

- no solution (does not exist)
- one solution (exists and is unique)
- infinitely many solutions (exists, not unique)

Augmented matrix:

$$\left(\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right) \leftarrow \text{same as Expl 3}$$

Reduced echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

x_3, x_4: free

x_1, x_2, x_5: basic

- No row of the form $(0\,0 \cdots 0 \mid b)$, $b \neq 0$
  So there is at least one solution.

- There are free variables.
  So there are $\boxed{\text{infinitely many solutions}}$